Quantile Correlations and Quantile Autoregressive Modeling

Guodong Li, Yang Li, and Chih-Ling Tsai

In this article, we propose two important measures, quantile correlation (QCOR) and quantile partial correlation (QPCOR). We then apply them to quantile autoregressive (QAR) models, and introduce two valuable quantities, the quantile autocorrelation function (QACF) and the quantile partial autocorrelation function (QPACF). This allows us to extend the Box–Jenkins three-stage procedure (model identification, model parameter estimation, and model diagnostic checking) from classical autoregressive models to quantile autoregressive models. Specifically, the QPACF of an observed time series can be employed to identify the autoregressive order, while the QACF of residuals obtained from the fitted model can be used to assess the model adequacy. We not only demonstrate the asymptotic properties of QCOR and QPCOR, but also show the large sample results of QACF, QPACF, and the quantile version of the Box–Pierce test. Moreover, we obtain the bootstrap approximations to the distributions of parameter estimators and proposed measures. Simulation studies indicate that the proposed methods perform well in finite samples, and an empirical example is presented to illustrate usefulness. Supplementary materials for this article are available online.

KEY WORDS: Autocorrelation function; Bootstrap method; Box-Jenkins method; Quantile partial correlation.

1. INTRODUCTION

In the last decade, quantile regression has attracted considerable attention. There are two major reasons for such popularity. The first is that quantile regression estimation (Koenker and Bassett 1978) can be robust to non-Gaussian or heavy-tailed data, and it includes the commonly used least absolute deviation (LAD) method as a special case. The second is that the quantile regression model allows practitioners to provide more easily interpretable regression estimates obtained via quantiles $\tau \in (0, 1)$. More references about quantile regression estimation and interpretation can be found in the seminal book of Koenker (2005). Further extensions of quantile regression to various model and data structures can be found in the literature, for example, Machado and Silva (2005) for count data, Mu and He (2007) for power transformed data, Peng and Huang (2008) and Wang and Wang (2009) for survival analysis, He and Liang (2000) and Wei and Carroll (2009) for regression with measurement error, Ando and Tsay (2011) for regression with augmented factors, and Kai, Li, and Zou (2011) for semiparametric varying-coefficient partially linear models, among others.

In addition to the regression context, the quantile technique has been applied to the field of time series; see, for example, Koul and Saleh (1995) and Cai, Stander, and Davies (2012) for autoregressive (AR) models, Ling and McAleer (2004) for unstable AR models, and Xiao and Koenker (2009) for generalized autoregressive conditional heteroscedastic (GARCH) models. In particular, Koenker and Xiao (2006) established important statistical properties for quantile autoregressive (QAR)

models, which expanded the classical AR model into a new area. In AR models, Box-Jenkins three-stage procedure (i.e., model identification, model parameter estimation, and model diagnostic checking) has been commonly used for the last 40 years. This motivates us to extend the classical Box-Jenkins approach from AR to QAR models. In the classical AR model, it is known that model identification usually relies on the partial autocorrelation function (PACF) of the observed time series, while model diagnosis commonly depends on the autocorrelation function (ACF) of model residuals. Detailed illustrations of model identification and diagnosis can be found in Box, Jenkins, and Reinsel (2008). The aim of this article is to introduce two novel measures to examine the linear and partially linear relationships between any two random variables for a given quantile $\tau \in (0, 1)$. We name them quantile correlation (QCOR) and quantile partial correlation (QPCOR). Based on these two measures, we propose the quantile partial autocorrelation function (QPACF) and the quantile autocorrelation function (QACF) to identify the order of the QAR model and to assess model adequacy, respectively. We also employ the bootstrap approach to study the performance of QPACF and QACF. It is noteworthy that the application of QCOR and QPCOR is not limited to QAR models. They can be used as broadly as the classical correlation and partial correlation measures in various contexts.

The rest of this article is organized as follows. Section 2 introduces QCOR and QPCOR. Furthermore, the asymptotic properties of their sample estimators are established. In Section 3, QPACF and its large sample property for identifying the order of QAR models are demonstrated. Subsequently, QACF and its resulting test statistics, together with their asymptotic properties, are provided to examine the model adequacy. The properties of QPACF and QACF, in conjunction with Koenker and Xiao's (2006) estimation results, lead us to propose a modified three-stage procedure for QAR models. Moreover, bootstrap approximations to the distributions of parameter estimators, the

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QPACF measure, and the QACF measure are studied. Section 4 conducts simulation experiments to assess the finite sample performance of the proposed methods, and Section 5 presents an empirical example to demonstrate their usefulness. Finally, we conclude the article with a brief discussion in Section 6. All technical proofs of lemmas and theorems are relegated to the Appendix.

2. CORRELATIONS

2.1 Quantile Correlation and Quantile Partial Correlation

For random variables X and Y, let $Q_{\tau,Y}$ be the τth unconditional quantile of Y and $Q_{\tau,Y}(X)$ be the τth quantile of Y conditional on X. One can show that $Q_{\tau,Y}(X)$ is independent of X, that is, $Q_{\tau,Y}(X) = Q_{\tau,Y}$ with probability one, if and only if the random variables $I(Y - Q_{\tau,Y} > 0)$ and X are independent, where $I(\cdot)$ is the indicator function. This fact has been used by He and Zhu (2003) and Mu and He (2007), and it also motivates us to define the quantile covariance given below. For $0 < \tau < 1$, define

$$\text{qcov}_\tau\{Y, X\} = \text{cov}\{I(Y - Q_{\tau,Y} > 0), X\}$$
$$= E\{\psi_\tau(Y - Q_{\tau,Y})(X - EX)\},$$

where the function $\psi_\tau(w) = \tau - I(w < 0)$. Note that $\text{qcov}_\tau\{Y, X\} = -\text{cov}\{F_{Y|X}(Q_{\tau,Y}), X\}$, where $F_{Y|X}(\cdot)$ is the cumulative distribution of Y conditional on X. Subsequently, the quantile correlation can be defined as follows,

$$\text{qcor}_\tau\{Y, X\} = \frac{\text{qcov}_\tau\{Y, X\}}{\sqrt{\text{var}\{\psi_\tau(Y - Q_{\tau,Y})\}\text{var}(X)}}$$
$$= \frac{E\{\psi_\tau(Y - Q_{\tau,Y})(X - EX)\}}{\sqrt{(\tau - \tau^2)\sigma_X^2}}, \quad (2.1)$$

where $\sigma_X^2 = \text{var}(X)$. Accordingly, $-1 \leq \text{qcor}_\tau\{Y, X\} \leq 1$.

In the simple linear regression with the quadratic loss function, there is a nice relationship between the slope and correlation. Hence, it is of interest to find a connection between the quantile slope and $\text{qcor}_\tau\{Y, X\}$. To this end, consider a simple quantile linear regression,

$$(a_0, b_0) = \underset{a,b}{\text{argmin}}\, E[\rho_\tau(Y - a - bX)],$$

in which one attempts to approximate $Q_{\tau,Y}(X)$ by a linear function $a_0 + b_0X$ (see Koenker 2005), where $\rho_\tau(w) = w[\tau - I(w < 0)]$, $a_0 = Q_{\tau,Y-b_0X}$, and b_0 is the quantile slope. Let $\varepsilon = Y - a_0 - b_0X$, and we then obtain the relationship between b_0 and $\text{qcov}_\tau\{Y, X\}$ given below.

Lemma 1. Suppose that random variables X and ε have a joint density and $EX^2 < \infty$. Then, $\text{qcov}_\tau\{Y, X\} = \text{qcov}_\tau\{b_0X + \varepsilon, X\} = \varrho(b_0)$, where $\varrho(b) = E[\psi_\tau(\varepsilon - Q_{\tau,\varepsilon+bX} + bX)X]$ is a continuous and increasing function. In addition, $\varrho(b) = 0$ if and only if $b = 0$.

From the above lemma, $\text{qcov}_\tau\{Y, X\}$ is a rescaled version of the quantile slope b_0 via the function $\varrho(\cdot)$, and the slope $b_0 = 0$ if and only if $\text{qcov}_\tau\{Y, X\} = 0$. In addition, the relationship between $\text{qcor}_\tau\{Y, X\}$ and quantile slope can be obtained from Lemma 1 straightforwardly. Moreover, it implies that the quantile correlation increases with the quantile slope. As the classical correlation coefficient, $\text{qcor}_\tau\{Y, X\}$ lies between -1 and 1 and

it is a unit-free measure. However, the range of quantile slope is not bounded. Hence, it is natural to employ the quantile correlation rather than the slope to rank the significance of predictors on the quantile of Y.

It is noteworthy that the proposed quantile covariance here does not enjoy the symmetry property of the classical covariance, that is, $\text{qcov}_\tau(Y, X) \neq \text{qcov}_\tau(X, Y)$. This is because the first argument of the quantile covariance or the quantile correlation is related to the τth quantile, while the second argument is the same as that of the classical covariance. Accordingly, $\text{qcor}_\tau(Y, X) \neq \text{qcor}_\tau(X, Y)$. It is also of interest to find that Blomqvist (1950) introduced a measure to study the dependence between two random variables, which has the form of $\text{cov}\{I(Y > Q_{0.5,Y}), I(X > Q_{0.5,X})\}$. He further linked his measure to the Kendall's rank correlation (see also Cox and Hinkley 1974, p. 204). Under specific conditions with $\tau = 0.5$, we can find the relationship between Blomqvist's measure and our proposed measure. In other words, let random variables X and Y be standardized so that $Q_{0.5,X} = Q_{0.5,Y} = 0$ and $E|X| = 1$. Moreover, assume that $|X|$ is independent of $\text{sgn}(X)$ and $\text{sgn}(Y)$, where $\text{sgn}(w)$ is the sign of w. We then have $\text{qcov}_{0.5}\{Y, X\} = 0.5\text{cov}\{\text{sgn}(Y), \text{sgn}(X)|X|\} = 2\text{cov}\{I(Y > 0), I(X > 0)\}$. It is of interest to note that, for daily return series in financial markets, the independence of $|X|$ and $\text{sgn}(X)$ is a stylized fact (Ryden, Terasvirta, and Asbrink 1998).

Suppose that a quantile linear regression model has the response Y, a $q \times 1$ vector of covariates $\mathbf{Z}$, and an additional covariate X. In the classical regression model, one can construct the partial correlation to measure the linear relationship between variables Y and X after adjusting for (or controlling for) vector $\mathbf{Z}$ (e.g., see Chatterjee and Hadi 2006). This motivates us to propose the quantile partial correlation function. To this end, let

$$(\alpha_1, \beta_1') = \underset{\alpha,\beta}{\text{argmin}}\, E(X - \alpha - \beta'\mathbf{Z})^2,$$

where $(\alpha, \beta')'$ is a vector of unknown parameters. Accordingly, $\alpha_1 + \beta_1'\mathbf{Z}$ is the linear effect of $\mathbf{Z}$ on X. Next, consider

$$(\alpha_2, \beta_2') = \underset{\alpha,\beta}{\text{argmin}}\, E[\rho_\tau(Y - \alpha - \beta'\mathbf{Z})].$$

As a result, $\alpha_2 + \beta_2'\mathbf{Z}$ is the linear effect of $\mathbf{Z}$ on the τth quantile of Y (i.e., the linear approximation of $Q_{\tau,Y}(\mathbf{Z})$). It can also be shown that $E(X - \alpha_1 - \beta_1'\mathbf{Z}) = 0$, $E[\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})] = 0$, $E[\mathbf{Z}\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})] = \mathbf{0}$, and the values of α_1, β_1, α_2, and β_2 are unique if the random vector $(Y, X, \mathbf{Z}')'$ has a joint density with $EX^2 < \infty$ and $E\|\mathbf{Z}\|^2 < \infty$, where $\mathbf{0}$ is a $q \times 1$ vector with all elements being zero. Using these facts, we define the quantile partial correlation as follows,

$$\text{qpcor}_\tau\{Y, X|\mathbf{Z}\}$$
$$= \frac{\text{cov}\{\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z}), X - \alpha_1 - \beta_1'\mathbf{Z}\}}{\sqrt{\text{var}\{\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})\}\text{var}\{X - \alpha_1 - \beta_1'\mathbf{Z}\}}}$$
$$= \frac{E[\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})(X - \alpha_1 - \beta_1'\mathbf{Z})]}{\sqrt{(\tau - \tau^2)E(X - \alpha_1 - \beta_1'\mathbf{Z})^2}}$$
$$= \frac{E[\psi_\tau(Y - \alpha_2 - \beta_2'\mathbf{Z})X]}{\sqrt{(\tau - \tau^2)\sigma_{X|\mathbf{Z}}^2}}, \quad (2.2)$$

where $\sigma_{X|\mathbf{Z}}^2 = E(X - \alpha_1 - \beta_1'\mathbf{Z})^2$. By treating $Y - \alpha_2 - \beta_2'\mathbf{Z}$ as a new response variable and X as a covariate, we can then

apply Lemma 1 to obtain the relationship between the resulting quantile regression slope and $\text{qpcor}_\tau\{Y, X|\mathbf{Z}\}$.

2.2 Sample Quantile Correlation and Sample Quantile Partial Correlation

Suppose that the data $\{(Y_i, X_i, \mathbf{Z}_i')', i = 1, \ldots, n\}$ are identically and independently generated from a distribution of $(Y, X, \mathbf{Z}')'$. Let $\widehat{Q}_{\tau,Y} = \inf\{y : F_n(y) \geq \tau\}$ be the sample τth quantile of $Y_1, \ldots, Y_n$, where $F_n(y) = n^{-1}\sum_{i=1}^n I(Y_i \leq y)$ is the empirical distribution function. Based on Equation (2.1), the sample estimate of the quantile correlation $\text{qcor}_\tau\{Y, X\}$ is defined as

$$\widehat{\text{qcor}}_\tau\{Y, X\} = \frac{1}{\sqrt{(\tau - \tau^2)\widehat{\sigma}_X^2}} \cdot \frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}),$$

(2.3)

where $\bar{X} = n^{-1}\sum_{i=1}^n X_i$ and $\widehat{\sigma}_X^2 = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

To study the asymptotic property of $\widehat{\text{qcor}}_\tau\{Y, X\}$, denote $f_Y(\cdot)$ and $f_{Y|X}(\cdot)$ as the density of Y and the conditional density of Y given X, respectively. In addition, let $\mu_X = E(X)$, $\mu_{X|Y} = E[f_{Y|X}(Q_{\tau,Y})X]/f_Y(Q_{\tau,Y})$, $\Sigma_{11} = E(X - \mu_X)^4 - \sigma_X^4$,

$$\Sigma_{12} = E[\psi_\tau(Y - Q_{\tau,Y})(X - \mu_{X|Y})]^2 - [\text{qcor}_\tau\{Y, X\}]^2,$$
$$\Sigma_{13} = E[\psi_\tau(Y - Q_{\tau,Y})(X - \mu_{X|Y})(X - \mu_X)^2] - \sigma_X^2 \cdot \text{qcov}_\tau\{Y, X\},$$

and

$$\Omega_1 = \frac{1}{\tau - \tau^2}\left[\frac{\Sigma_{11}(\text{qcov}_\tau\{Y, X\})^2}{4\sigma_X^6} - \frac{\Sigma_{13} \cdot \text{qcov}_\tau\{Y, X\}}{\sigma_X^4} + \frac{\Sigma_{12}}{\sigma_X^2}\right],$$

where σ_X^2 is defined as in the previous subsection. Then, we obtain the following result.

Theorem 1. Suppose that $E(X^4) < \infty$, there exists a $\pi > 0$ such that the conditional density $f_{Y|X}(\cdot)$ is uniformly integrable on $[Q_{\tau,Y} - \pi, Q_{\tau,Y} + \pi]$, and the density $f_Y(\cdot)$ is continuous and positive. Then $\sqrt{n}\left(\widehat{\text{qcor}}_\tau\{Y, X\} - \text{qcor}_\tau\{Y, X\}\right) \to_d N(0, \Omega_1)$.

To apply the above theorem, one needs to estimate the asymptotic variance Ω_1. To this end, we employ a nonparametric approach, such as the Nadaraya–Watson regression, to estimate the function $m(y) = E(X|Y = y)$, and denote the estimator by $\widehat{m}(y)$. We further assume that the random vector (X, Y) has a joint density, and then it can be shown that $\mu_{X|Y} = E(X|Y = Q_{\tau,Y})$. Accordingly, we obtain the estimate, $\widehat{\mu}_{X|Y} = \widehat{m}(\widehat{Q}_{\tau,Y})$, where $\widehat{Q}_{\tau,Y}$ is the τth sample quantile of $\{Y_1, \ldots, Y_n\}$. Finally, the rest of the quantities contained in Ω_1, including μ_X, σ_X^2, $\text{qcov}_\tau\{Y, X\}$, Σ_{11}, Σ_{12}, and Σ_{13}, can be, respectively, estimated by $\widehat{\mu}_X = \bar{X} = n^{-1}\sum_{i=1}^n X_i$, $\widehat{\sigma}_X^2 = n^{-1}\sum_{i=1}^n (X_i - \widehat{\mu}_X)^2$, $\widehat{\text{qcov}}_\tau\{Y, X\} = n^{-1}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X})$, $\widehat{\Sigma}_{11} = n^{-1}\sum_{i=1}^n (X_i - \widehat{\mu}_X)^4 - \widehat{\sigma}_X^4$, $\widehat{\Sigma}_{12} = n^{-1}\sum_{i=1}^n [\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \widehat{\mu}_{X|Y})]^2 - [\widehat{\text{qcov}}_\tau\{Y, X\}]^2$, and $\widehat{\Sigma}_{13} = n^{-1}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \widehat{\mu}_{X|Y})(X_i - \widehat{\mu}_X)^2 - \widehat{\sigma}_X^2\widehat{\text{qcov}}_\tau\{Y, X\}$. As a result, we obtain an estimate of Ω_1, and denote it by $\widehat{\Omega}_1$.

We next estimate the quantile partial correlation $\text{qpcor}_\tau\{Y, X\}$. Let

$$(\widehat{\alpha}_1, \widehat{\beta}_1') = \underset{\alpha,\beta}{\text{argmin}} \sum_{i=1}^n (X_i - \alpha - \beta'\mathbf{Z}_i)^2 \quad \text{and}$$

$$(\widehat{\alpha}_2, \widehat{\beta}_2') = \underset{\alpha,\beta}{\text{argmin}} \sum_{i=1}^n \rho_\tau(Y_i - \alpha - \beta'\mathbf{Z}_i).$$

Based on Equation (2.2), the sample quantile partial correlation is defined as

$$\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\} = \frac{1}{\sqrt{(\tau - \tau^2)\widehat{\sigma}_{X|\mathbf{Z}}^2}} \cdot \frac{1}{n}\sum_{i=1}^n \psi_\tau(Y_i - \widehat{\alpha}_2 - \widehat{\beta}_2'\mathbf{Z}_i)X_i, \quad (2.4)$$

where $\widehat{\sigma}_{X|\mathbf{Z}}^2 = n^{-1}\sum_{i=1}^n (X_i - \widehat{\alpha}_1 - \widehat{\beta}_1'\mathbf{Z}_i)^2$.

To investigate the asymptotic property of $\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\}$, denote the conditional density of Y given $\mathbf{Z}$ and the conditional density of Y given $\mathbf{Z}$ and X by $f_{Y|\mathbf{Z}}(\cdot)$ and $f_{Y|\mathbf{Z},X}(\cdot)$, respectively. In addition, let $\theta_1 = (\alpha_1, \beta_1')'$, $\theta_2 = (\alpha_2, \beta_2')'$, $\mathbf{Z}^* = (1, \mathbf{Z}')'$, $\Sigma_{21} = E[f_{Y|\mathbf{Z},X}(\theta_2'\mathbf{Z}^*)X\mathbf{Z}^*]$, $\Sigma_{22} = E[f_{Y|\mathbf{Z}}(\theta_2'\mathbf{Z}^*)\mathbf{Z}^*\mathbf{Z}^{*\prime}]$, $\Sigma_{20} = \Sigma_{21}'\Sigma_{22}^{-1}$, $\Sigma_{23} = E(X - \theta_1'\mathbf{Z}^*)^4 - \sigma_{X|\mathbf{Z}}^4$,

$$\Sigma_{24} = E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)(X - \Sigma_{20}\mathbf{Z}^*)]^2 - \{E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X]\}^2,$$
$$\Sigma_{25} = E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)(X - \Sigma_{20}\mathbf{Z}^*)(X - \theta_1'\mathbf{Z}^*)^2] - \sigma_{X|\mathbf{Z}}^2 \cdot E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X],$$

and

$$\Omega_2 = \frac{1}{\tau - \tau^2}\left[\frac{\Sigma_{23}(E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X])^2}{4\sigma_{X|\mathbf{Z}}^6} - \frac{\Sigma_{25} \cdot E[\psi_\tau(Y - \theta_2'\mathbf{Z}^*)X]}{\sigma_{X|\mathbf{Z}}^4} + \frac{\Sigma_{24}}{\sigma_{X|\mathbf{Z}}^2}\right],$$

where α_1, β_1, α_2, β_2, and $\sigma_{X|\mathbf{Z}}^2$ are defined as in the previous section. Then, we have the following result.

Theorem 2. Suppose that Σ_{21} and Σ_{22} are finite, $EX^4 < \infty$, $E\|\mathbf{Z}\|^4 < \infty$, Σ_{22} and $E(\mathbf{Z}^*\mathbf{Z}^{*\prime})$ are positive definite matrices, and there exists a $\pi > 0$ such that $f_{Y|\mathbf{Z}}(\theta_2'\mathbf{Z}^* + \cdot)$ and $f_{Y|\mathbf{Z},X}(\theta_2'\mathbf{Z}^* + \cdot)$ are uniformly integrable on $[-\pi, \pi]$. Then $\sqrt{n}[\widehat{\text{qpcor}}_\tau\{Y, X|\mathbf{Z}\} - \text{qpcor}_\tau\{Y, X|\mathbf{Z}\}] \to_d N(0, \Omega_2)$.

Let $e^* = Y - \theta_2'\mathbf{Z}^*$, and assume that the random vector $(e^*, X, \mathbf{Z}')'$ has the joint density $f_{e^*,\mathbf{Z},X}(\cdot, \cdot, \cdot)$. Denote the marginal density of e^* and the conditional density of e^* given $\mathbf{Z}$ and X by $f_{e^*}(\cdot)$ and $f_{e^*|\mathbf{Z},X}(\cdot)$, respectively. It can be verified that

$$\Sigma_{21} = E[f_{e^*|\mathbf{Z},X}(0)X\mathbf{Z}^*] = \int\int f_{e^*,\mathbf{Z},X}(0, \mathbf{z}, x)x\mathbf{z}^*dxd\mathbf{z}$$
$$= f_{e^*}(0)\int\int \frac{f_{e^*,\mathbf{Z},X}(0, \mathbf{z}, x)}{f_{e^*}(0)}x\mathbf{z}^*dxd\mathbf{z}$$
$$= f_{e^*}(0) \cdot E[X\mathbf{Z}^*|e^* = 0],$$

$\Sigma_{22} = f_{e^*}(0) \cdot E[\mathbf{Z}^*\mathbf{Z}^{*\prime}|e^* = 0]$, and $\Sigma_{20} = E[X\mathbf{Z}^{*\prime}|e^* = 0]\{E[\mathbf{Z}^*\mathbf{Z}^{*\prime}|e^* = 0]\}^{-1}$, where $\mathbf{z}^* = (1, \mathbf{z}')'$. Hence, the conditional densities $f_{Y|\mathbf{Z}}(\cdot)$ and $f_{Y|\mathbf{Z},X}(\cdot)$ can be replaced by

conditional expectations on one random variable e^* only. To obtain the estimate of Σ_{20}, we first calculate the quantile regression estimate, $\widehat{\theta}_2 = (\widehat{\alpha}_2, \widehat{\beta}'_2)'$, and then compute its resulting quantile residuals, $\widehat{e}_i^* = Y_i - \widehat{\theta}'_2 \mathbf{Z}_i^*$ for $i = 1, \ldots, n$. Applying the same nonparametric technique as that used for estimating $\mu_{X|Y}$ in Theorem 1, for any given $e^* = e_g^*$, we can estimate each of the vector and matrix components in $\mathbf{m}_1(e_g^*) = E[X\mathbf{Z}^*|e^* = e_g^*]$ and $\mathbf{m}_2(e_g^*) = E[\mathbf{Z}^*\mathbf{Z}^{*\prime}|e^* = e_g^*]$, respectively, from the data $\{(\widehat{e}_i^*, X_i, \mathbf{Z}_i') = (Y_i - \widehat{\theta}'_2 \mathbf{Z}_i^*, X_i, \mathbf{Z}_i'), i = 1, \ldots, n\}$. This yields the estimate $\widehat{\mathbf{m}}'_1(e_g^*)[\widehat{\mathbf{m}}_2(e_g^*)]^{-1}$. Accordingly, we have $\widehat{\Sigma}_{20} = \widehat{\Sigma}'_{21} \widehat{\Sigma}_{22}^{-1} = \widehat{\mathbf{m}}'_1(0)[\widehat{\mathbf{m}}_2(0)]^{-1}$. Under some regularity conditions, we can show that $\widehat{\Sigma}_{20}$ is a consistent estimator of Σ_{20}.

Subsequently, the rest of the quantities involved in Ω_2, namely $\sigma_{X|\mathbf{Z}}^2$, $\mathrm{qcov}_\tau\{e^*, X\}$, Σ_{23}, Σ_{24}, and Σ_{25}, can be, respectively, estimated by $\widehat{\sigma}_{X|\mathbf{Z}}^2 = n^{-1} \sum_{i=1}^n (X_i - \widehat{\alpha}_1 - \widehat{\beta}'_1 \mathbf{Z}_i)^2$, $\widehat{\mathrm{qcov}}_\tau \{e^*, X\} = n^{-1} \sum_{i=1}^n \psi_\tau(Y_i - \widehat{\theta}'_2 \mathbf{Z}_i^*) X_i$, $\widehat{\Sigma}_{23} = n^{-1} \sum_{i=1}^n (X_i - \widehat{\theta}'_1 \mathbf{Z}_i^*)^4 - \widehat{\sigma}_{X|\mathbf{Z}}^4$, $\widehat{\Sigma}_{24} = n^{-1} \sum_{i=1}^n [\psi_\tau(Y_i - \widehat{\theta}_2 \mathbf{Z}_i^*)(X_i - \widehat{\Sigma}_{20}\mathbf{Z}_i^*)]^2 - [\widehat{\mathrm{qcov}}_\tau\{e^*, X\}]^2$, and $\widehat{\Sigma}_{25} = n^{-1} \sum_{i=1}^n \psi_\tau(Y_i - \widehat{\theta}_2 \mathbf{Z}_i^*)(X_i - \widehat{\Sigma}_{20}\mathbf{Z}_i^*)(X_i - \widehat{\theta}'_1 \mathbf{Z}_i^*)^2 - \widehat{\sigma}_{X|\mathbf{Z}}^2 \cdot \widehat{\mathrm{qcov}}_\tau\{e^*, X\}$. Consequently, we obtain the estimate of Ω_2, and denote it by $\widehat{\Omega}_2$. We next apply the quantile correlation and quantile partial correlation to quantile autoregressive models.

3. QUANTILE AUTOREGRESSIVE ANALYSIS

Suppose that $\{y_t\}$ is a strictly stationary and ergodic time series, and $\mathcal{F}_t$ is the σ-field generated by $\{y_t, y_{t-1}, \ldots\}$. We then follow the approach of Koenker and Xiao (2006) to define QAR models; that is, conditional on $\mathcal{F}_{t-1}$, the τth quantile of y_t has the form

$$Q_\tau(y_t|\mathcal{F}_{t-1}) = \phi_0(\tau) + \phi_1(\tau)y_{t-1} + \cdots + \phi_p(\tau)y_{t-p}$$
$$\text{for} \quad 0 < \tau < 1, \qquad (3.1)$$

where the $\phi_i(\cdot)$'s are unknown functions mapping from $[0, 1] \to R$. Note that the right-hand side of (3.1) is monotonically increasing in τ. Let $\{u_t\}$ be iid Uniform(0, 1) random variables, and then model (3.1) can be rewritten as

$$y_t = \phi_0(u_t) + \phi_1(u_t)y_{t-1} + \cdots + \phi_p(u_t)y_{t-p}.$$

For simplicity, the QAR model in this article refers to Equation (3.1) with a strictly stationary and ergodic time series $\{y_t\}$.

For this QAR model, Koenker and Xiao (2006) derived the asymptotic distributions of the estimators of the $\phi_i(\cdot)$'s. Hence, this section mainly introduces the QPACF of a time series to identify the order of a QAR model, and then uses the QACF of residuals to assess the adequacy of the fitted model. To present the theoretical results of the proposed correlation measures given in this section, let $\Rightarrow$ denote weak convergence on D, where $D = D[0, 1]$ is the space of functions on $[0, 1]$ endowed with the Skorohod topology (Billingsley 1999).

3.1 The QPACF of Time Series

For the positive integer k, let $\mathbf{z}_{t,k-1} = (y_{t-1}, \ldots, y_{t-k+1})'$, $(\alpha_1, \beta'_1) = \mathrm{argmin}_{\alpha, \beta} E(y_{t-k} - \alpha - \beta' \mathbf{z}_{t,k-1})^2$, and $(\alpha_{2,\tau}, \beta'_{2,\tau}) = \mathrm{argmin}_{\alpha, \beta} E[\rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,k-1})]$, where the notation (α_1, β'_1) is a slight abuse since they have been used to denote the regression parameters in Section 2, and the notation

$(\alpha_{2,\tau}, \beta'_{2,\tau})$ is to emphasize its dependence on τ. From Equation (2.2), we obtain the quantile partial correlation between y_t and y_{t-k} after adjusting for the linear effect of $\mathbf{z}_{t,k-1}$,

$$\phi_{kk,\tau} = \mathrm{qpcor}_\tau\{y_t, y_{t-k}|\mathbf{z}_{t,k-1}\}$$
$$= \frac{E[\psi_\tau(y_t - \alpha_{2,\tau} - \beta'_{2,\tau}\mathbf{z}_{t,k-1})y_{t-k}]}{\sqrt{(\tau - \tau^2)E(y_{t-k} - \alpha_1 - \beta'_1\mathbf{z}_{t,k-1})^2}},$$

and it is independent of the time index t due to the strict stationarity of $\{y_t\}$. Analogously to the definition of the classical PACF (Fan and Yao 2003, chap. 2), we name $\phi_{kk,\tau}$ the QPACF of the time series $\{y_t\}$. It is also noteworthy that $\phi_{11,\tau} = \mathrm{qcor}_\tau\{y_t, y_{t-1}\}$. We next show the cut-off property of QPACF.

Lemma 2. Suppose that y_t has a conditional density on the σ-field $\mathcal{F}_{t-1}$ and $Ey_t^2 < \infty$. If $\phi_p(\tau) \neq 0$ with $p > 0$, then $\phi_{pp,\tau} \neq 0$ and $\phi_{kk,\tau} = 0$ for $k > p$.

The above lemma indicates that the proposed QPACF plays the same role as that of the PACF in the classical AR model identification. Furthermore, define the conditional quantile error,

$$e_{t,\tau} = y_t - \phi_0(\tau) - \phi_1(\tau)y_{t-1} - \cdots - \phi_p(\tau)y_{t-p}. \quad (3.2)$$

By (3.1), the random variable $I(e_{t,\tau} > 0)$ is independent of y_{t-k} for any $k > 0$, and $(\alpha_{2,\tau}, \beta'_{2,\tau}) = (\phi_0(\tau), \phi_1(\tau), \ldots, \phi_p(\tau), 0, \ldots, 0)$ for $k > p$.

In practice, one needs the sample estimate of QPACF. To this end, let

$$(\widetilde{\alpha}_1, \widetilde{\beta}'_1) = \mathrm{argmin}_{\alpha, \beta} \sum_{t=k+1}^n (y_{t-k} - \alpha - \beta' \mathbf{z}_{t,k-1})^2,$$

$$(\widetilde{\alpha}_{2,\tau}, \widetilde{\beta}'_{2,\tau}) = \mathrm{argmin}_{\alpha, \beta} \sum_{t=k+1}^n \rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,k-1}),$$

and $\widetilde{\sigma}_{y|\mathbf{z}}^2 = n^{-1} \sum_{t=k+1}^n (y_{t-k} - \widetilde{\alpha}_1 - \widetilde{\beta}'_1\mathbf{z}_{t,k-1})^2$. According to (2.4), we obtain the estimation for $\phi_{kk,\tau}$,

$$\widetilde{\phi}_{kk,\tau} = \frac{1}{\sqrt{(\tau - \tau^2)\widetilde{\sigma}_{y|\mathbf{z}}^2}} \cdot \frac{1}{n} \sum_{t=k+1}^n \psi_\tau(y_t - \widetilde{\alpha}_{2,\tau} - \widetilde{\beta}'_{2,\tau}\mathbf{z}_{t,k-1})y_{t-k},$$

and we name it the sample QPACF of the time series.

To study the asymptotic property of $\widetilde{\phi}_{kk,\tau}$, we introduce the following assumption, which is similar to Condition A.3 in Koenker and Xiao (2006).

Assumption 1. $Ey_t^2 < \infty$, $E[y_t - E(y_t|\mathcal{F}_{t-1})]^2 > 0$, and $f_{t-1}(\cdot)$ is uniformly integrable on $\mathcal{U}$, where $f_{t-1}(\cdot)$ is the conditional density of $e_{t,\tau}$ on the σ-field $\mathcal{F}_{t-1}$, $\mathcal{U} = \{u : 0 < F(u) < 1\}$ and $F(\cdot)$ is the marginal distribution of $e_{t,\tau}$.

Let $\mathbf{z}_{t,k-1}^* = (1, \mathbf{z}'_{t,k-1})' = (1, y_{t-1}, \ldots, y_{t-k+1})'$. Moreover, let $A_0 = E[y_{t-k}\mathbf{z}_{t,k-1}^*]$, $A_1(\tau) = E[f_{t-1}(0)y_{t-k}\mathbf{z}_{t,k-1}^*]$, $\Sigma_{30} = E[\mathbf{z}_{t,k-1}^*\mathbf{z}_{t,k-1}^{*\prime}]$, $\Sigma_{31}(\tau) = E[f_{t-1}(0)\mathbf{z}_{t,k-1}^*\mathbf{z}_{t,k-1}^{*\prime}]$,

$$\Sigma_{32}(\tau_1, \tau_2) = E(y_t^2) - A'_1(\tau_1)\Sigma_{31}^{-1}(\tau_1)A_0 - A'_1(\tau_2)\Sigma_{31}^{-1}(\tau_2)A_0$$
$$+ A'_1(\tau_1)\Sigma_{31}^{-1}(\tau_1)\Sigma_{30}\Sigma_{31}^{-1}(\tau_2)A_1(\tau_2),$$

and

$$\Omega_3(\tau_1, \tau_2) = \frac{E[\psi_{\tau_1}(e_{t,\tau_1})\psi_{\tau_2}(e_{t,\tau_2})]\Sigma_{32}(\tau_1, \tau_2)}{\sqrt{(\tau_1 - \tau_1^2)(\tau_2 - \tau_2^2)}E(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2}.$$

Then, we obtain the asymptotic result given below.

Theorem 3. Suppose that, for each $\tau \in \mathcal{I}$, $A_1(\tau)$, and $\Sigma_{31}(\tau)$ are finite, and $\Sigma_{31}(\tau)$ is a positive definite matrix, where $\mathcal{I} \subset (0, 1)$ is a closed interval. If Assumption 1 is satisfied and $k > p$, then $\sqrt{n}\widetilde{\phi}_{kk,\tau} \Rightarrow B_1(\tau)$ for all $\tau \in \mathcal{I}$, where $B_1(\tau)$ is a Gaussian process with mean zero and covariance kernel $\Omega_3(\tau_1, \tau_2) = E[B_1(\tau_1)B_1(\tau_2)]$ for $\tau_1, \tau_2 \in \mathcal{I}$.

When the conditional quantile errors $\{e_{t,\tau}\}$ are iid, the random variable $e_{t,\tau} - E(e_{t,\tau})$ can be shown to be independent of τ, so we define $e_t = e_{t,\tau} - E(e_{t,\tau})$ for simplicity. Note that $e_{t,\tau} = e_t - Q_{\tau,e_t}$ with $E(e_t) = 0$. Accordingly, $f_{t-1}(0) = f(Q_{\tau,e_t})$ and $\Sigma_{31}(\tau) = E[f_{t-1}(0)\mathbf{z}_{t,k-1}^*\mathbf{z}_{t,k-1}^{*'}] = f(Q_{\tau,e_t})E[\mathbf{z}_{t,k-1}^*\mathbf{z}_{t,k-1}^{*'}]$, where $f(\cdot)$ is the density function of e_t. By the condition that $\Sigma_{31}(\tau)$ is a positive definite matrix, we have that $f(Q_{\tau,e_t}) > 0$ for all $\tau \in \mathcal{I}$. In addition, the finite matrix assumption of $\Sigma_{31}(\tau)$ leads to $f(Q_{\tau,e_t}) < \infty$ for all $\tau \in \mathcal{I}$. As a result, $0 < f_{t-1}(0) < \infty$.

For a given $\tau \in \mathcal{I}$, $\sqrt{n}\widetilde{\phi}_{kk,\tau} \to_d N\{0, \Omega_3(\tau, \tau)\}$. To estimate the asymptotic variance, we first apply the Hendricks and Koenker (1992) method to obtain the estimation of $f_{t-1}(0)$ given below

$$\widetilde{f}_{t-1}(0) = \frac{2h}{\widetilde{Q}_{\tau+h}(y_t|\mathcal{F}_{t-1}) - \widetilde{Q}_{\tau-h}(y_t|\mathcal{F}_{t-1})},$$

where $\widetilde{Q}_\tau(y_t|\mathcal{F}_{t-1}) = \widetilde{\phi}_0(\tau) + \widetilde{\phi}_1(\tau)y_{t-1} + \cdots + \widetilde{\phi}_k(\tau)y_{t-k}$ is the estimated τth quantile of y_t and h is the bandwidth selected via appropriate methods (e.g., see Koenker and Xiao 2006). Afterward, we can use sample averaging to approximate A_0, $A_1(\tau)$, Σ_{30}, $\Sigma_{31}(\tau)$, $E(y_t^2)$, and $E(y_{t-k} - \alpha_1 - \beta_1'\mathbf{z}_{t,k-1})^2$ by replacing $f_{t-1}(0)$, α_1, and β_1 in those quantities, respectively, with $\widetilde{f}_{t-1}(0)$, $\widetilde{\alpha}_1$ and $\widetilde{\beta}_1$. Accordingly, we obtain an estimate of $\Omega_3(\tau, \tau)$, and denote it $\widehat{\Omega}_3$. In sum, we are able to use the threshold values $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$ to check the significance of $\widetilde{\phi}_{kk,\tau}$. To demonstrate how to use the above theorem to identify the order of a QAR model, we generate the observations $y_1, \ldots, y_{200}$ from $y_t = \Phi^{-1}(u_t) + a(u_t)y_{t-1}$, where Φ is the standard normal cumulative distribution function, $a(x) = \max\{0.8 - 1.6x, 0\}$, and $\{u_t\}$ is an iid sequence with uniform distribution on $[0, 1]$. We attempt to fit the QAR model (3.1) with $\tau = 0.2, 0.4, 0.6,$ and 0.8, respectively, to the observed data $\{y_t\}$. Figure 1 presents the sample QPACF $\widetilde{\phi}_{kk,\tau}$ for each τ with the reference lines $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$. We may conclude that the order p is 1 when $\tau = 0.2$ and 0.4, while p is 0 when $\tau = 0.6$ and 0.8.
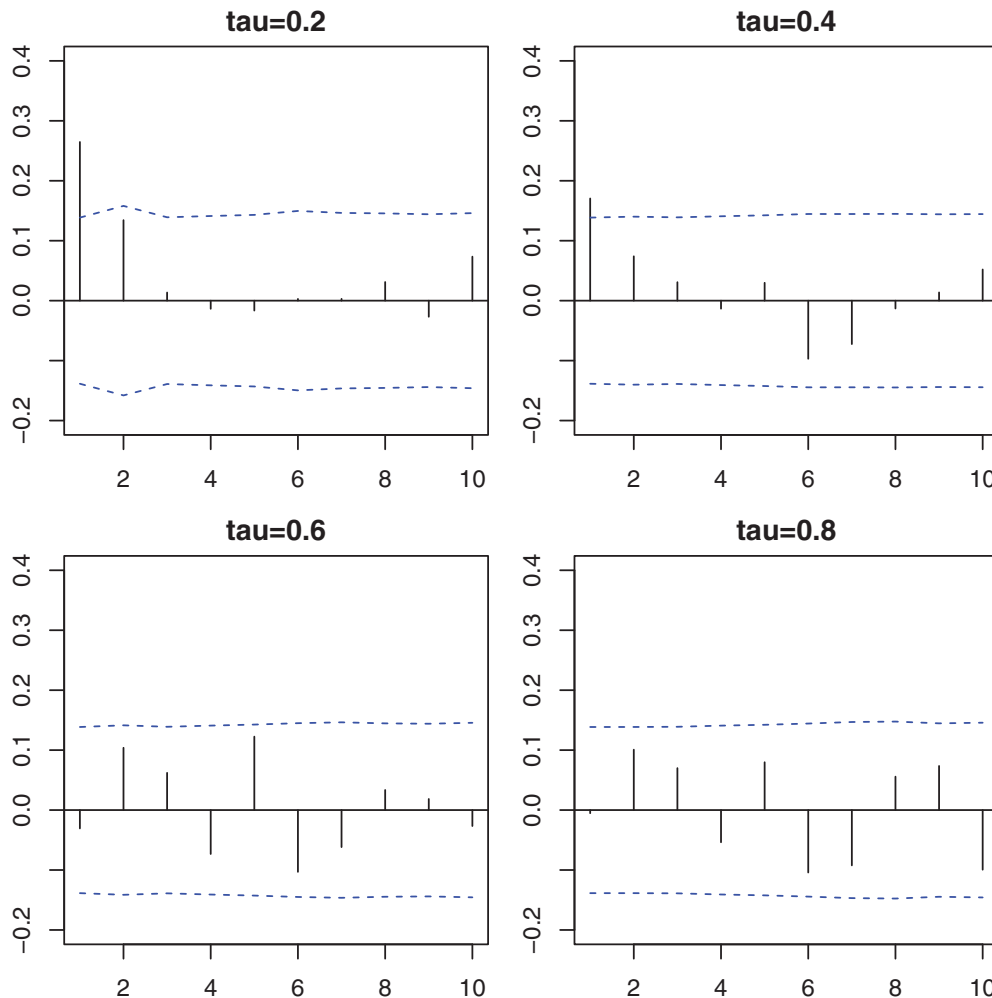

Figure 1. The sample QPACF of the observed time series, $\widetilde{\phi}_{kk,\tau}$, with $\tau = 0.2, 0.4, 0.6,$ and 0.8. The dashed lines correspond to $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$.

3.2 Parameter Estimation and the QACF of Residuals

From the results of QPACF in the previous section, the order p of model (3.1) can be identified, and we then assume it is known a priori. We subsequently fit the data with the QAR(p) model to obtain parameter estimates and their asymptotic properties. Let $\boldsymbol{\phi} = (\phi_0, \phi_1, \ldots, \phi_p)'$ be the parameter vector in model (3.1) and $\boldsymbol{\phi}(\tau) = (\phi_0(\tau), \phi_1(\tau), \ldots, \phi_p(\tau))'$ be the true value of $\boldsymbol{\phi}$. It is noteworthy that $(\alpha_{2,\tau}, \beta'_{2,\tau})'$ defined in Section 3.1 is $\boldsymbol{\phi}(\tau)$ when $k = p$. Consider

$$\widetilde{\boldsymbol{\phi}}(\tau) = \arg\min_{\boldsymbol{\phi}} \sum_{t=p+1}^{n} \rho_\tau(y_t - \boldsymbol{\phi}' \mathbf{z}^*_{t,p}),$$

where $\mathbf{z}^*_{t,p} = (1, \mathbf{z}'_{t,p})' = (1, y_{t-1}, \ldots, y_{t-p})'$. In addition, let $\Sigma_{40} = E[\mathbf{z}^*_{t,p} \mathbf{z}^{*\prime}_{t,p}]$, $\Sigma_{41}(\tau) = E[f_{t-1}(0) \mathbf{z}^*_{t,p} \mathbf{z}^{*\prime}_{t,p}]$, and $\Omega_4(\tau_1, \tau_2) = \sqrt{(\tau_1 - \tau_1^2)(\tau_2 - \tau_2^2)} \Sigma_{41}^{-1}(\tau_1) \Sigma_{40} \Sigma_{41}^{-1}(\tau_2)$. Suppose that, for each $\tau \in \mathcal{I}$, $\Sigma_{41}(\tau)$ is a finite and positive definite matrix, and Assumption 1 is satisfied. By Theorem 2 in Koenker and Xiao (2006), we obtain that

$$\sqrt{n}\{\widetilde{\boldsymbol{\phi}}(\tau) - \boldsymbol{\phi}(\tau)\} \Rightarrow B_2(\tau) \quad \text{for all} \quad \tau \in \mathcal{I}, \tag{3.3}$$

where $B_2(\tau)$ is a Gaussian process with mean zero and covariance kernel $\Omega_4(\tau_1, \tau_2) = E[B_2(\tau_1)B_2(\tau_2)]$ for $\tau_1, \tau_2 \in \mathcal{I}$.

Suppose that $\{e_{t,\tau}\}$ are iid random errors with $e_{t,\tau} = e_t - Q_{\tau, e_t}$ and $E(e_t) = 0$. We can then apply the same techniques as those discussed earlier after Theorem 3 to show that the positive definite matrix condition of $\Sigma_{41}(\tau)$ implies $f_{t-1}(0) = f(Q_{\tau, e_t}) > 0$ for all $\tau \in \mathcal{I}$. In addition, the finite matrix assumption of $\Sigma_{41}(\tau)$ leads to $f_{t-1}(0) = f(Q_{\tau, e_t}) < \infty$ for all $\tau \in \mathcal{I}$. We next construct diagnostic tests to assess the adequacy of the fitted model.

For the errors $\{e_{t,\tau}\}$ defined in (3.2), we employ Equation (2.1) and the fact that $Q_{\tau, e_{t,\tau}} = 0$, and obtain QACF of $\{e_{t,\tau}\}$ as follows,

$$\rho_{k,\tau} = \frac{E\{\psi_\tau(e_{t,\tau})[e_{t-k,\tau} - E(e_{t,\tau})]\}}{\sqrt{(\tau - \tau^2)\sigma^2_{e,\tau}}},$$

where $\sigma^2_{e,\tau} = \text{var}(e_{t,\tau})$. We can show that $\rho_{k,\tau} = 0$ for $k > 0$. Hence, we are able to use $\rho_{k,\tau}$ to assess the model fit. In the sample version, we consider the residuals of the QAR model,

$$\widetilde{e}_{t,\tau} = y_t - \widetilde{\phi}_0(\tau) - \widetilde{\phi}_1(\tau)y_{t-1} - \cdots - \widetilde{\phi}_p(\tau)y_{t-p},$$

for $t = p+1, \ldots, n$, and $\widetilde{e}_{t,\tau} = 0$ for $t = 1, \ldots, p$. It can be verified that the τth empirical quantile of $\{\widetilde{e}_{t,\tau}\}$ is zero. Based on this fact and Equation (2.3), we obtain the estimation of $\rho_{k,\tau}$,

$$r_{k,\tau} = \frac{1}{\sqrt{(\tau - \tau^2)\widetilde{\sigma}^2_{e,\tau}}} \cdot \frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(\widetilde{e}_{t,\tau})(\widetilde{e}_{t-k,\tau} - \widetilde{\mu}_{e,\tau}),$$

where k is a positive integer, $\widetilde{\mu}_{e,\tau} = n^{-1} \sum_{t=k+1}^{n} \widetilde{e}_{t,\tau}$ and $\widetilde{\sigma}^2_{e,\tau} = n^{-1} \sum_{t=k+1}^{n} (\widetilde{e}_{t,\tau} - \widetilde{\mu}_{e,\tau})^2$. We name $r_{k,\tau}$ the sample QACF of residuals.

Adapting the classical linear time series approach (Li 2004), we examine the significance of $\{r_{k,\tau}\}$ individually and jointly. For the given positive integer K, let $\mathbf{e}_{t-1,K} = (e_{t-1,\tau}, \ldots, e_{t-K,\tau})'$, $\Sigma_{50} = E[\mathbf{e}_{t-1,K} \mathbf{z}^{*\prime}_{t,p}]$, $\Sigma_{51}(\tau) =$

$E[f_{t-1}(0)\mathbf{e}_{t-1,K} \mathbf{z}^{*\prime}_{t,p}]$,

$$\Sigma_{52}(\tau_1, \tau_2)$$
$$= E(\mathbf{e}_{t-1,K} \mathbf{e}'_{t-1,K}) + \Sigma_{51}(\tau_1)\Sigma_{41}^{-1}(\tau_1)\Sigma_{40}\Sigma_{41}^{-1}(\tau_2)\Sigma'_{51}(\tau_2)$$
$$- \Sigma_{51}(\tau_1)\Sigma_{41}^{-1}(\tau_1)\Sigma'_{50} - \Sigma_{50}\Sigma_{41}^{-1}(\tau_2)\Sigma'_{51}(\tau_2),$$

and

$$\Omega_5(\tau_1, \tau_2) = \frac{E[\psi_{\tau_1}(e_{t,\tau_1})\psi_{\tau_2}(e_{t,\tau_2})]\Sigma_{52}(\tau_1, \tau_2)}{\sqrt{(\tau_1 - \tau_1^2)(\tau_2 - \tau_2^2)\sigma^2_{e,\tau_1}\sigma^2_{e,\tau_2}}}.$$

Then, we obtain the asymptotic property of $R_\tau = (r_{1,\tau}, \ldots, r_{K,\tau})'$ given below.

Theorem 4. Suppose that, for each $\tau \in \mathcal{I}$, $\Sigma_{41}(\tau)$ and $\Sigma_{51}(\tau)$ are finite, and $\Sigma_{41}(\tau)$ is a positive definite matrix, where $\mathcal{I}$ is defined as in Theorem 3. If Assumption 1 holds and the order p of model (3.1) is correctly identified, then $\sqrt{n}R_\tau \Rightarrow \mathbf{B}_3(\tau)$ for all $\tau \in \mathcal{I}$, where $\mathbf{B}_3(\tau)$ is a K-dimensional Gaussian process with mean zero and covariance kernel $\Omega_5(\tau_1, \tau_2) = E[\mathbf{B}_3(\tau_1)\mathbf{B}'_3(\tau_2)]$ for $\tau_1, \tau_2 \in \mathcal{I}$.

For a given $\tau \in \mathcal{I}$, $\sqrt{n}R_\tau \to_d N\{0, \Omega_5(\tau, \tau)\}$. Applying the same techniques as used in the estimate of $\Omega_3(\tau, \tau)$, we are able to estimate the asymptotic variance $\Omega_5(\tau, \tau)$ and denote it $\widehat{\Omega}_5$. In addition, let the kth diagonal element of $\widehat{\Omega}_5$ be $\widehat{\Omega}_{5k}$. Then, one can employ $r_{k,\tau}/\sqrt{\widehat{\Omega}_{5k}}$ to examine the significance of the kth lag in the residual series.

To check the significance of R_τ jointly, it is natural to consider the test statistic $R'_\tau \widehat{\Omega}_5^{-1} R_\tau$. When $\{e_{t,\tau}\}$ is an iid sequence, the matrix $\Omega_5(\tau, \tau) = I_K - \sigma^{-2}_{e,\tau}\Sigma_{50}\Sigma_{40}^{-1}\Sigma'_{50}$ has a rank of $K - p$. This motivates us to consider a Box–Pierce type test statistic (Box and Pierce 1970),

$$Q_{\text{BP}}(K) = n \sum_{j=1}^{K} r^2_{j,\tau} = nR'_\tau R_\tau \to_d \mathbf{B}'_3(\tau)\mathbf{B}_3(\tau),$$

where $\mathbf{B}'_3(\tau)\mathbf{B}_3(\tau)$ can be approximated by a χ^2_{K-p} distribution when the errors $\{e_{t,\tau}\}$ are iid random variables. In the case of non-iid random errors, the following procedure can be used to calculate the critical value:

(i) Generate a random vector $\boldsymbol{\zeta}_1 = (\zeta_{11}, \ldots, \zeta_{1K})'$ from a standard multivariate normal distribution, and calculate the value of $\text{BP}_1 = \sum_{i=1}^{K} \lambda_i \xi^2_{1i}$, where the λ_i's are the eigenvalues of $\widehat{\Omega}_5$.

(ii) Repeat Step (i) $M - 1$ times by generating independent standard multivariate normal random vectors $\boldsymbol{\zeta}_2, \ldots, \boldsymbol{\zeta}_M$, and then calculate the values of $\text{BP}_2, \ldots, \text{BP}_M$.

(iii) Obtain the empirical $100(1 - \alpha)$th percentile of $\{\text{BP}_1, \ldots, \text{BP}_M\}$, and use it as the critical value at the α level of significance.

Accordingly, one can employ $Q_{\text{BP}}(K)$ to test the significance of $(\rho_{1,\tau}, \ldots, \rho_{K,\tau})'$ jointly.

3.3 Modified Three-Stage Procedure

To apply the above theoretical results, we adopt the Box–Jenkins three-stage procedure and propose the following modified version for QAR models.

1. Model identification: Choose K a priori to be the largest lag order in a set of candidate models. Then, employ the QPACF of the observed series with Theorem 3 to select the tentative QAR model (namely QAR(p)). Accordingly, the sample QPACF has a cutoff after lag p.

2. Model estimation: Estimate the tentative model in the first stage as well as the backward selection models in the third stage given below.

3. Model selection and diagnosis:

   3a. From the QAR(p) model, employ Koenker and Xiao's (2006) Theorem 2 (see also Equation (3.3)) to conduct p tests of the null hypotheses $H_0 : \phi_j(\tau) = 0$ for $1 \leq j \leq p$. Then, remove the lag with the largest p-value that is greater than the predetermined significance level, say 5%.

   3b. Repeat Step 3a to remove nonsignificant lags sequentially, until all lags remaining in the model are significant.

   3c. Employ the Box–Pierce type test, $Q_{\mathrm{BP}}(K)$, to check the adequacy of the resulting model, and apply the Wald test, $W_n(\tau)$, in Koenker and Xiao (2006) to assess whether those removed lags are jointly insignificant. If either of these two tests fails, then add the last lag removed in Step 3b back into the model.

   3d. Repeat Step 3c until there exists a model passing both $Q_{\mathrm{BP}}(K)$ and $W_n(\tau)$ tests.

   3e. If no model can be found in Step 3d, then try a larger p (or K), or transform the data, or consider alternative model structures.

Remark 1. In the first stage, one may use the Bayesian information criterion (BIC) of Schwarz (1978) or the Akaike information criterion (AIC) of Akaike (1973) to identify the order of the lag. Since the theoretical properties of AIC and BIC in the QAR model have not been established yet, both AIC and BIC can be viewed as supplementary guidelines to assist in the model-selection process as suggested by Box, Jenkins, and Reinsel (2008, p. 212). In Step 3c of the third stage, one can use the test computed from the sample QACF of residuals to examine the significance of $(\rho_{1,\tau}, \ldots, \rho_{K,\tau})$ individually (see Theorem 4). In practice, the rejection of the individual test may occur even in random series (see Box, Jenkins, and Reinsel 2008, p. 341). In addition, a few lags with marginal significance obtained from the individual test are not likely to affect the conclusion of the Box–Pierce type test for assessing the joint effect across all K lags. Hence, we recommend the $Q_{\mathrm{BP}}(K)$ test in Step 3c, and the individual test can be used as an auxiliary tool. Moreover, we include the $W_n(\tau)$ test to examine the impact of removed lags. In sum, Step 3c mainly focuses on the joint effect of model fitting, which provides clear guidance for finding a final model.

Remark 2. It is noteworthy that our theoretical results are based on model (3.1), which satisfies quantile monotonicity. Hence, one needs to check for possible crossings among the fitted quantile functions in practical applications. To examine crossings, we suggest plotting the fitted quantile functions over the entire quantile region. If they do not intersect each other, then crossing is not a serious issue. We may also apply an informal test (such as the binomial test) of the null hypothesis

$H_0 : p^* = p_0^*$ via the observed proportion of crossings from any two given quantile functions in $n - 1$ segments, where p_0^* is a prespecified proportion of crossings. Based on our limited experience, we suggest that $0.001 \leq p_0^* \leq 0.01$, while practitioners can choose a very small p_0^* for large sample sizes to examine quantile crossing. For example, when testing $p_0^* = 0.001$ in 1000 observations, more than two crossing points would yield a warning message, since the resulting p-value of the binomial test is less than 0.05. If there exists strong evidence of crossing, one may follow Koenker and Xiao's (2006) suggestion to transform the vector of variables, $\mathbf{z}_{t,p}^* = (1, y_{t-1}, \ldots, y_{t-p})'$. An alternative approach is to consider the dynamic additive quantile models (see Gourieroux and Jasiak 2008) or the constrained QAR models adopted from Bondell, Reich, and Wang (2010), which can avoid quantile crossing.

3.4 Bootstrap Approximations

To conduct model identification, parameter estimation, and model diagnostic checking, we need to estimate the variances of $\Omega_3(\tau, \tau)$, $\Omega_4(\tau, \tau)$, and $\Omega_5(\tau, \tau)$, respectively. Since this quantity involves the nonparametric estimate of the density function $f_{t-1}(0)$, it is essential to employ the bootstrap approach to investigate the performance of the proposed three-stage procedure for QAR models. In the context of quantile regression models, several bootstrap methods have been proposed; see, for example, He and Hu (2002) and Kocherginsky, He, and Mu (2005). It is noteworthy that $f_{t-1}(0)$ often depends on past observations, so the above methods may not be directly applicable to QAR models. Hence, we consider a bootstrap approach from Rao and Zhao (1992) by introducing a series of random weights to the loss function, see also Jin, Ying, and Wei (2001), Feng, He, and Hu (2011), and Li, Leng, and Tsai (2013).

Let $\{\omega_t\}$ be iid nonnegative random variables with mean 1 and variance 1. To approximate the distributions of $\widetilde{\phi}_{kk,\tau}$ in Theorem 3, we first calculate the weighted quantile estimator of $(\alpha_{2,\tau}, \beta_{2,\tau}')$ with weights $\{\omega_t\}$,

$$(\widetilde{\alpha}_{2,\tau}^*, \widetilde{\beta}_{2,\tau}^{*\prime}) = \underset{\alpha, \beta}{\operatorname{argmin}} \sum_{t=k+1}^{n} \omega_t \rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,k-1}),$$

and then obtain the weighted QPACF

$$\widetilde{\phi}_{kk,\tau}^* = \frac{1}{\sqrt{(\tau - \tau^2)\widetilde{\sigma}_{y|\mathbf{z}}^2}}$$
$$\cdot \frac{1}{n} \sum_{t=k+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\alpha}_{2,\tau}^* - \widetilde{\beta}_{2,\tau}^{*\prime} \mathbf{z}_{t,k-1}) y_{t-k}.$$

For the asymptotic distributions of $\widetilde{\phi}(\tau)$ at (3.3) and R_τ in Theorem 4, we consider the weighted quantile estimator of $\phi(\tau)$,

$$\widetilde{\phi}^*(\tau) = \underset{\alpha, \beta}{\operatorname{argmin}} \sum_{t=p+1}^{n} \omega_t \rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,p}),$$

and calculate a weighted sample QACF

$$r_{k,\tau}^* = \frac{1}{\sqrt{(\tau - \tau^2)\widetilde{\sigma}_{e,\tau}^2}} \cdot \frac{1}{n} \sum_{t=\max\{k,p\}+1}^{n} \omega_t \psi_\tau(y_t - \widetilde{\phi}^{*\prime}(\tau) \mathbf{z}_{t,p}^*)$$
$$\times (y_{t-k} - \widetilde{\phi}^{*\prime}(\tau) \mathbf{z}_{t-k,p}^*),$$

where $\mathbf{z}_{t,p}^* = (1, \mathbf{z}_{t,p}')'$. Let $R_\tau^* = (r_{1,\tau}^*, \ldots, r_{K,\tau}^*)'$, and the theoretical properties of $\widetilde{\phi}_{kk,\tau}^*$, $\widetilde{\boldsymbol{\phi}}^*(\tau)$, and R_τ^* are given below.

Theorem 5. Under the assumptions of Theorems 3 and 4, it holds that, conditional on $y_1, \ldots, y_n$,

(a) $\sqrt{n}(\widetilde{\phi}_{kk,\tau}^* - \widetilde{\phi}_{kk,\tau}) \Rightarrow B_1^*(\tau)$,

(b) $\sqrt{n}\{\widetilde{\boldsymbol{\phi}}^*(\tau) - \widetilde{\boldsymbol{\phi}}(\tau)\} \Rightarrow B_2^*(\tau)$,

(c) $\sqrt{n}(R_\tau^* - R_\tau) \Rightarrow \mathbf{B}_3^*(\tau)$,

for all $\tau \in \mathcal{I}$, where $B_1^*(\tau)$, $B_2^*(\tau)$, and $\mathbf{B}_3^*(\tau)$ are Gaussian processes with mean zero and the same covariance kernels as in Theorem 3, (3.3), and Theorem 4, respectively.

The above theorem allows us to approximate the distributions of $\widetilde{\phi}_{kk,\tau}, \widetilde{\boldsymbol{\phi}}(\tau), r_{k,\tau}$, and R_τ via their corresponding bootstrap analogues for the QAR analysis of model identification, parameter estimation, and model diagnostic checking. Hence, this method can avoid the numerical problems encountered in computing the estimated asymptotic variances in Theorem 3, Equation (3.3), and Theorem 4. The detailed bootstrap algorithms and theoretical proofs can be found in the supplementary material.

4. SIMULATION STUDIES

This section investigates the finite sample performance of the proposed measures and tests in Section 3. In all experiments, we conduct 1000 realizations for each combination of sample sizes $n = 100, 200$, and 500 and quantiles, $\tau = 0.25, 0.50$, and 0.75. In addition, the number of bootstrapped samples is set to $B = 1000$, and the random weights $\{\omega_t\}$ follow the standard exponential distribution. Moreover, we present the bias (BIAS), sample standard deviation (SSD), and empirical coverage probability of the proposed quantile measure (or estimate) across 1000 realizations.

In this simulation study, we generate the data from the following process,

$$y_t = 0.3y_{t-1} + 0.3v_t I(v_t > \chi_{0.35}^2)y_{t-2} + v_t, \qquad (4.1)$$

where $\{v_t\}$ are iid chi-squared random variables with one degree of freedom, and χ_α^2 is the αth quantile of v_t such that $P(v_t < \chi_\alpha^2) = \alpha$. It is noteworthy that $\{y_t\}$ is a nonnegative time series, $Q_\tau(y_t|\mathcal{F}_{t-1}) = \chi_\tau^2 + 0.3y_{t-1}$ for $\tau \leq 0.35$, and $Q_\tau(y_t|\mathcal{F}_{t-1}) = \chi_\tau^2 + 0.3y_{t-1} + 0.3\chi_\tau^2 y_{t-2}$ for $\tau > 0.35$. In other words, the resulting series is QAR(1) when $\tau \leq 0.35$, while it is QAR(2) when $\tau > 0.35$. Accordingly, the conditional quantile errors, $e_{t,\tau} = y_t - Q_\tau(y_t|\mathcal{F}_{t-1})$, depend on y_{t-2}, which are not iid random variables.

We employ the approach of Hendricks and Koenker (1992) to estimate the density function, $f_{t-1}(0)$, with the two bandwidth selection methods proposed by Bofinger (1975) and Hall and Sheather (1988), respectively, which are given below.

$$h_B = n^{-1/5}\left\{\frac{4.5\phi^4(\Phi^{-1}(\tau))}{[2(\Phi^{-1}(\tau))^2 + 1]^2}\right\}^{1/5} \quad \text{and}$$

$$h_{HS} = n^{-1/3}z_\alpha^{2/3}\left\{\frac{1.5\phi^2(\Phi^{-1}(\tau))}{2(\Phi^{-1}(\tau))^2 + 1}\right\}^{1/3},$$

where $\phi(\cdot)$ is the standard normal density function, $z_\alpha = \Phi^{-1}(1 - \alpha/2)$ for the construction of $1 - \alpha$ confidence intervals, and α is set to 0.05. Furthermore, we consider two more

Table 1. Bias (BIAS), sample standard deviation (SSD), and empirical coverage probability at the 95% nominal level of $\widetilde{\phi}_{kk,\tau}$, computed from the time series data with non-iid conditional quantile errors, at lags $k = 2, 3$, and 4 for $\tau = 0.25$, and $k = 3$ and 4 for $\tau = 0.5$ and 0.75

				Empirical coverage probability				
n	k	BIAS	SSD	h_{HS}	h_B	$3h_{HS}$	$0.6h_B$	Boot
				$\tau = 0.25$				
100	2	−0.0226	0.1014	93.4	93.4	93.4	93.5	93.0
	3	−0.0160	0.1014	96.0	95.8	95.3	95.9	93.1
	4	−0.0144	0.1034	94.7	94.5	93.6	94.8	91.2
200	2	−0.0142	0.0727	93.9	94.1	94.0	94.3	93.1
	3	−0.0086	0.0713	94.5	94.5	94.5	94.4	92.2
	4	−0.0174	0.0755	93.8	93.2	93.1	93.8	92.3
500	2	−0.0087	0.0444	95.5	95.4	95.4	95.4	94.2
	3	−0.0013	0.0438	95.7	95.6	95.5	95.8	94.2
	4	−0.0037	0.0428	96.0	96.0	95.8	96.1	94.4
				$\tau = 0.5$				
100	3	−0.0109	0.1034	93.7	93.8	93.8	94.1	93.7
	4	−0.0267	0.1021	94.3	94.3	94.3	94.3	93.6
200	3	−0.0031	0.0729	95.2	94.9	95.4	94.9	94.1
	4	−0.0096	0.0710	94.5	94.3	94.4	94.5	94.5
500	3	−0.0029	0.0463	94.7	94.6	94.5	94.8	93.6
	4	−0.0050	0.0462	94.0	93.9	93.9	94.0	93.5
				$\tau = 0.75$				
100	3	−0.0068	0.1046	93.7	93.9	93.6	94.0	93.9
	4	−0.0185	0.0993	94.9	94.9	94.8	95.0	93.3
200	3	−0.0005	0.0736	94.6	94.7	94.8	94.9	94.0
	4	−0.0074	0.0723	94.0	94.2	94.1	94.1	93.6
500	3	0.0009	0.0443	95.4	95.4	95.4	95.5	95.1
	4	−0.0059	0.0453	94.6	94.7	94.7	94.6	93.9

bandwidths, $0.6h_B$ and $3h_{HS}$, suggested by Koenker and Xiao (2006). In sum, we have four bandwidth choices. This allows us to construct the confidence limits of $\widetilde{\phi}_{kk,\tau}$, $\widetilde{\boldsymbol{\phi}}(\tau)$, and $r_{k,\tau}$ by estimating the variance $\Omega_3(\tau, \tau)$ in Theorem 3, the variance $\Omega_4(\tau, \tau)$ in (3.3), and the variance $\Omega_5(\tau, \tau)$ in Theorem 4, respectively.

To understand the performance of QPACF in the first stage of model identification, Table 1 reports the biases, sample standard deviations, and empirical coverage probabilities at the 95% nominal level of $\widetilde{\phi}_{kk,\tau}$, at $k = 2, 3$, and 4 for $\tau = 0.25$, and at $k = 3$ and 4 for $\tau = 0.5$ and 0.75, respectively. The simulation results indicate that biases and sample standard deviations become smaller as the sample size gets larger, which is consistent with theoretical findings. In addition, empirical coverage probabilities calculated from the direct method via the asymptotic standard deviation are close to the nominal level, and all four bandwidths produce similar results. However, when the sample size is as small as 100 or 200, the direct method occasionally encounters a problem in computing the asymptotic variance (e.g., the inverse of a singular matrix or the square root of a negative value), which appears around 5 (or less) out of 1000 replications. Hence, we employ the bootstrap approach to calculate the empirical coverage probability. Table 1 shows that this approach performs well, although it is slightly inferior to the direct method.

Table 2. Bias (BIAS), sample standard deviation (SSD), and empirical coverage probability at the 95% nominal level of $\widetilde{\phi}_k(\tau)$, computed from the time series data with non-iid conditional quantile errors, at lags $k = 0, 1$ for $\tau = 0.25$ and $k = 0, 1$ and 2 for $\tau = 0.5$ and 0.75

				Empirical coverage probability			
n	k	BIAS	SSD	h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
				$\tau = 0.25$			
100	0	0.0042	0.0621	95.2	97.1	100.0	94.5
	1	0.0062	0.0254	95.3	95.3	99.3	94.5
200	0	−0.0007	0.0364	95.1	97.0	100.0	94.7
	1	0.0028	0.0140	95.1	95.5	99.3	94.6
500	0	−0.0015	0.0218	95.4	96.4	100.0	94.9
	1	0.0013	0.0071	95.4	95.0	99.2	95.2
				$\tau = 0.5$			
100	0	0.0782	0.2514	96.3	97.5	97.7	94.4
	1	0.0087	0.0814	89.1	90.0	90.3	90.8
	2	−0.0326	0.1236	88.5	91.3	90.5	91.8
200	0	0.0357	0.1777	94.7	96.1	100.0	94.9
	1	0.0043	0.0507	90.1	90.6	98.7	91.5
	2	−0.0134	0.0920	93.7	94.0	99.7	94.1
500	0	0.0095	0.1103	96.2	97.3	99.8	94.8
	1	0.0019	0.0314	90.3	92.1	96.6	92.3
	2	−0.0032	0.0610	95.2	96.2	99.3	94.9
				$\tau = 0.75$			
100	0	0.2244	0.5614	97.1	97.6	83.0	94.3
	1	−0.0001	0.1661	89.6	90.0	79.0	90.2
	2	−0.0855	0.2563	90.0	92.2	76.9	92.6
200	0	0.1071	0.3700	95.4	97.3	88.0	94.8
	1	0.0050	0.1122	90.7	91.3	84.2	91.5
	2	−0.0508	0.1900	92.2	93.1	85.7	93.3
500	0	0.0605	0.2344	95.4	96.5	92.7	95.2
	1	−0.0013	0.0722	91.1	91.4	88.5	92.8
	2	−0.0237	0.1257	92.1	95.8	90.5	94.3

Table 3. Bias (BIAS), sample standard deviation (SSD), and empirical coverage probability at the 95% nominal level of $r_{k,\tau}$, computed from the time series data with non-iid conditional quantile errors, at lags $k = 2, 4,$ and 6

				Empirical coverage probability			
n	QACF	BIAS	SSD	h_{HS}	h_B	$3h_{HS}$	$0.6h_B$
				$\tau = 0.25$			
100	$r_{2,\tau}$	−0.0137	0.0971	93.6	93.7	93.5	93.9
	$r_{4,\tau}$	0.0012	0.0981	95.8	95.8	95.7	95.8
	$r_{6,\tau}$	0.0044	0.0987	95.3	95.3	95.2	95.3
200	$r_{2,\tau}$	−0.0062	0.0680	95.5	95.6	95.5	95.5
	$r_{4,\tau}$	0.0008	0.0733	93.2	93.3	93.1	93.2
	$r_{6,\tau}$	0.0000	0.0702	95.3	95.1	95.1	95.3
500	$r_{2,\tau}$	−0.0022	0.0424	95.3	95.5	95.3	95.3
	$r_{4,\tau}$	0.0003	0.0455	94.6	94.6	94.6	94.6
	$r_{6,\tau}$	0.0002	0.0444	95.9	95.9	95.9	95.9
				$\tau = 0.5$			
100	$r_{2,\tau}$	0.0164	0.0545	78.9	78.8	78.9	78.9
	$r_{4,\tau}$	0.0014	0.0956	94.6	94.7	94.7	94.6
	$r_{6,\tau}$	−0.0019	0.0972	95.7	95.7	95.7	95.7
200	$r_{2,\tau}$	0.0084	0.0339	85.6	85.8	85.8	85.7
	$r_{4,\tau}$	0.0006	0.0677	94.9	94.8	95.0	94.8
	$r_{6,\tau}$	−0.0014	0.0697	95.1	95.1	95.2	95.1
500	$r_{2,\tau}$	0.0028	0.0186	94.8	94.8	94.9	94.7
	$r_{4,\tau}$	0.0000	0.0422	94.6	94.6	94.6	94.6
	$r_{6,\tau}$	0.0022	0.0450	95.1	95.1	95.1	95.1
				$\tau = 0.75$			
100	$r_{2,\tau}$	0.0100	0.0673	86.5	86.7	86.2	86.7
	$r_{4,\tau}$	−0.0068	0.0943	95.6	95.7	95.5	95.6
	$r_{6,\tau}$	−0.0067	0.0951	95.8	95.8	95.8	95.8
200	$r_{2,\tau}$	0.0057	0.0444	91.3	91.3	91.4	91.4
	$r_{4,\tau}$	−0.0042	0.0676	95.9	95.9	95.9	95.9
	$r_{6,\tau}$	−0.0030	0.0691	95.3	95.3	95.3	95.2
500	$r_{2,\tau}$	0.0037	0.0263	94.3	94.3	94.3	94.3
	$r_{4,\tau}$	0.0004	0.0429	94.8	94.7	94.7	94.7
	$r_{6,\tau}$	−0.0035	0.0440	95.1	95.0	95.0	95.0

We next examine the performance of parameter estimates $\widetilde{\phi}(\tau)$ in the second stage. Table 2 indicates that the biases and sample standard deviations decrease as the sample size increases. It is of interest to note that $\tau = 0.25$ often yields the best empirical coverage probability. This may be because the model fitting with $\tau = 0.25$ contains more observations than that with $\tau = 0.5$ and 0.75 in our simulation setting. In addition, the bandwidth $3h_{HS}$ performs worst, since it has the largest empirical coverage probabilities at $\tau = 0.25$ and 0.5 and the smallest empirical coverage probabilities at $\tau = 0.75$. This may result from having the largest bandwidth values over the whole range of the time period. Moreover, the other three bandwidths yield similar results.

We subsequently study the third stage of model diagnostics. According to model (4.1), we fit QAR(1) for $\tau = 0.25$ and QAR(2) for $\tau = 0.5$ and 0.75. Furthermore, the sample QACF of residuals are calculated at $K = 6$. Table 3 reports the biases, sample standard deviations, and empirical coverage probabilities at the 95% nominal level of $r_{k,\tau}$ at $k = 2, 4,$ and 6. Table 3 shows that biases and sample standard deviations become smaller as the sample size gets larger, which supports theoretical findings. In addition, the empirical coverage probabilities are close to the nominal level, except that $r_{2,\tau}$ (under $\tau = 0.5$

and 0.75) is near to the nominal level only for the sample size $n = 500$. This may be because conditional quantile errors depend on y_{t-2} in our simulation setting. Moreover, all four bandwidths yield similar results.

Finally, we examine the finite sample performance of the test statistic $Q_{BP}(K)$. To this end, we generate data from the following process,

$$y_t = 0.3y_{t-1} + 0.3\nu_t I(\nu_t > \chi^2_{0.35})y_{t-2} + \phi y_{t-3} + \nu_t,$$

where the ν_t are defined in (4.1). For simplicity, the QAR(2) model is employed for three quantiles. Note that $\phi = 0$ corresponds to the null hypothesis, while $\phi > 0$ is associated with the alternative hypothesis. The nominal level of the test is 5%, and the bandwidth is set to $0.6h_B$. Table 4 reports sizes and powers of $Q_{BP}(K)$ with $K = 6$. It shows that $Q_{BP}(K)$ controls the size well when n is large, and its power increases when the sample size or ϕ becomes larger. The other three bandwidths lead to similar findings, which are omitted.

In addition to the direct method (i.e., the nonbootstrap method) used in the second and third stages, we also employ

Table 4. Rejection rate of the test statistic $Q_{\mathrm{BP}}(K)$ with $K = 6$, $\tau = 0.25, 0.5,$ and $0.75,$ and the 5% nominal significance level

	$n = 100$			$n = 200$			$n = 500$		
ϕ	0.25	0.5	0.75	0.25	0.5	0.75	0.25	0.5	0.75
0.0	5.8	5.3	6.1	5.1	5.3	4.8	5.3	5.3	4.7
0.1	49.3	16.7	11.9	86.2	30.6	11.4	99.6	54.3	13.9
0.2	78.5	54.1	19.5	97.3	78.0	28.6	99.9	99.0	44.5

the bootstrap approach for studying the performance of parameter estimates and diagnostic measures. Although the bootstrap approach has theoretical justifications given in Section 3.4, its finite sample performance is usually not comparable to the direct method when the sample size is not large enough. Hence, we do not present the bootstrap results in Tables 2 and 3. Based on the above two simulation experiments, we suggest using the direct method in the modified three-stage procedure, together with the bandwidth $0.6h_B$ (also recommended by Koenker and Xiao 2006), for practical application. When the estimate of asymptotic variance of $\widetilde{\phi}_{kk,\tau}$ in Theorem 3 used at the first stage is not computable, one can consider the bootstrap approach. However, this does not exclude the possibility of using the bootstrap procedure when one encounters numerical problems in computing the estimated asymptotic variances in Equation (3.3) and Theorem 4.

We also conduct experiments to study the finite sample performance of the sample QCOR and the sample QPCOR in Section 2 as well as the proposed measures and tests in Section 3 under the assumption of iid conditional quantile errors. The simulation results are given in the online supplemental material.

5. NASDAQ COMPOSITE

This example considers the log return as a percentage of the daily closing price on the Nasdaq Composite from January 1, 2004 to December 31, 2007. There are 1006 observations in total, and Figure 2 depicts the time series plot and the classical sample ACF. It is not surprising that these returns (i.e., log returns) are uncorrelated and can be treated as evidence in support of the fair market theory. However, Veronesi (1999) found that stock markets underreact to good news in bad times and overreact to bad news in good times. Hence, Baur, Dimpfl, and Jung (2012) proposed aligning a good (bad) state with upper (lower) quantiles by fitting their stock returns data with the QAR(1) type models. This motivates us to employ the general QAR model along with our proposed techniques to explore the dependence pattern of stock returns at a lower quantile ($\tau = 0.05$), the median ($\tau = 0.5$), and an upper quantile ($\tau = 0.95$).

In this example, we follow the proposed procedure in Section 3.3 to find appropriate models. To this end, we choose $K = 15$ a priori to be the maximum lag in a family of candidate models. We first fit the returns at the lower quantile ($\tau = 0.05$). Panel A of Figure 3 presents the sample QPACF of the observed series, which indicates that lags 8 and 11 stand out. According to Theorem 3, the QAR(11) model is suggested. Subsequently, we refine the model via the backward variable selection procedure at the 5% significance level. As a result, lags {1, 8, 6, 7, 3, 5, 9} are removed sequentially, which leads to the following model,

$$\widehat{Q}_{0.05}(y_t|\mathcal{F}_{t-1}) = -0.7470_{0.0291} + 0.1176_{0.0508}\,y_{t-2}$$
$$+ 0.0809_{0.0583}\,y_{t-4} + 0.0739_{0.0507}\,y_{t-10}$$
$$+ 0.1258_{0.0546}\,y_{t-11}, \qquad (5.1)$$

where the subscripts of parameter estimates are their associated standard errors, and the bandwidth $0.6h_B$ is employed in this whole section. The p-value of the Wald test, $W_n(\tau)$, is 0.254, which implies that the deleted coefficients are jointly insignificant. In addition, although the sample QACF of residuals in Panel A shows that lags 2, 4, and 14 are marginally significant, the p-value of $Q_{\mathrm{BP}}(15)$ is 0.320 and it is not significant. It is worth noting that the coefficients at lags 4 and 10 in Equation (5.1) are significant at the 10% level, but not at the 5% level.
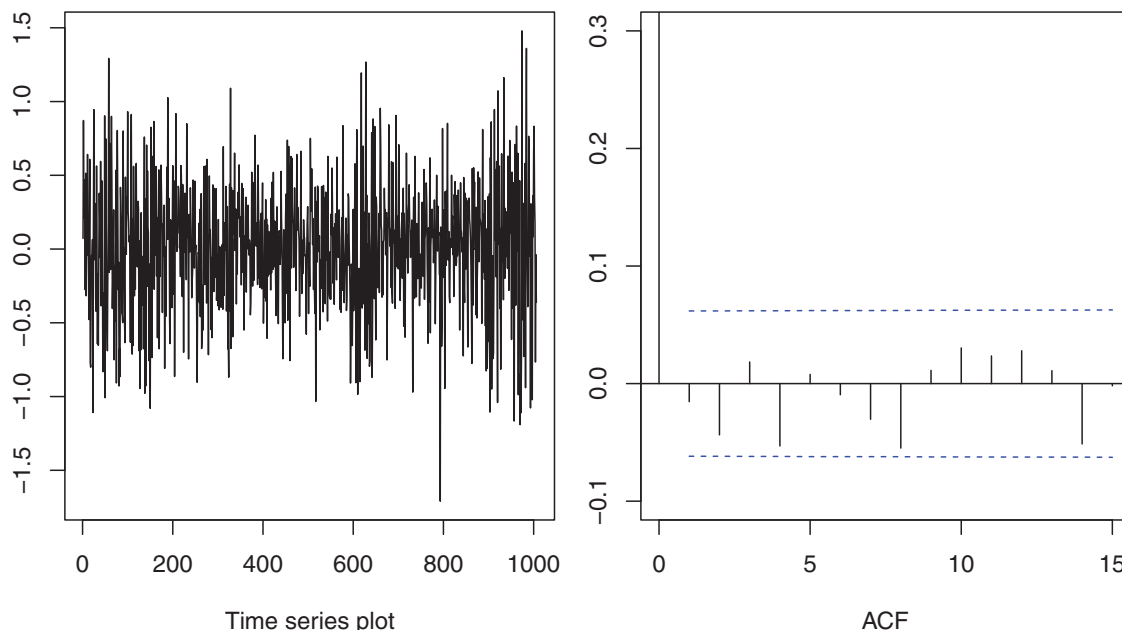

Figure 2. The time series plot and the sample ACF of the log return (as a percentage) of the daily closing price on the Nasdaq Composite from January 1, 2004 to December 31, 2007.
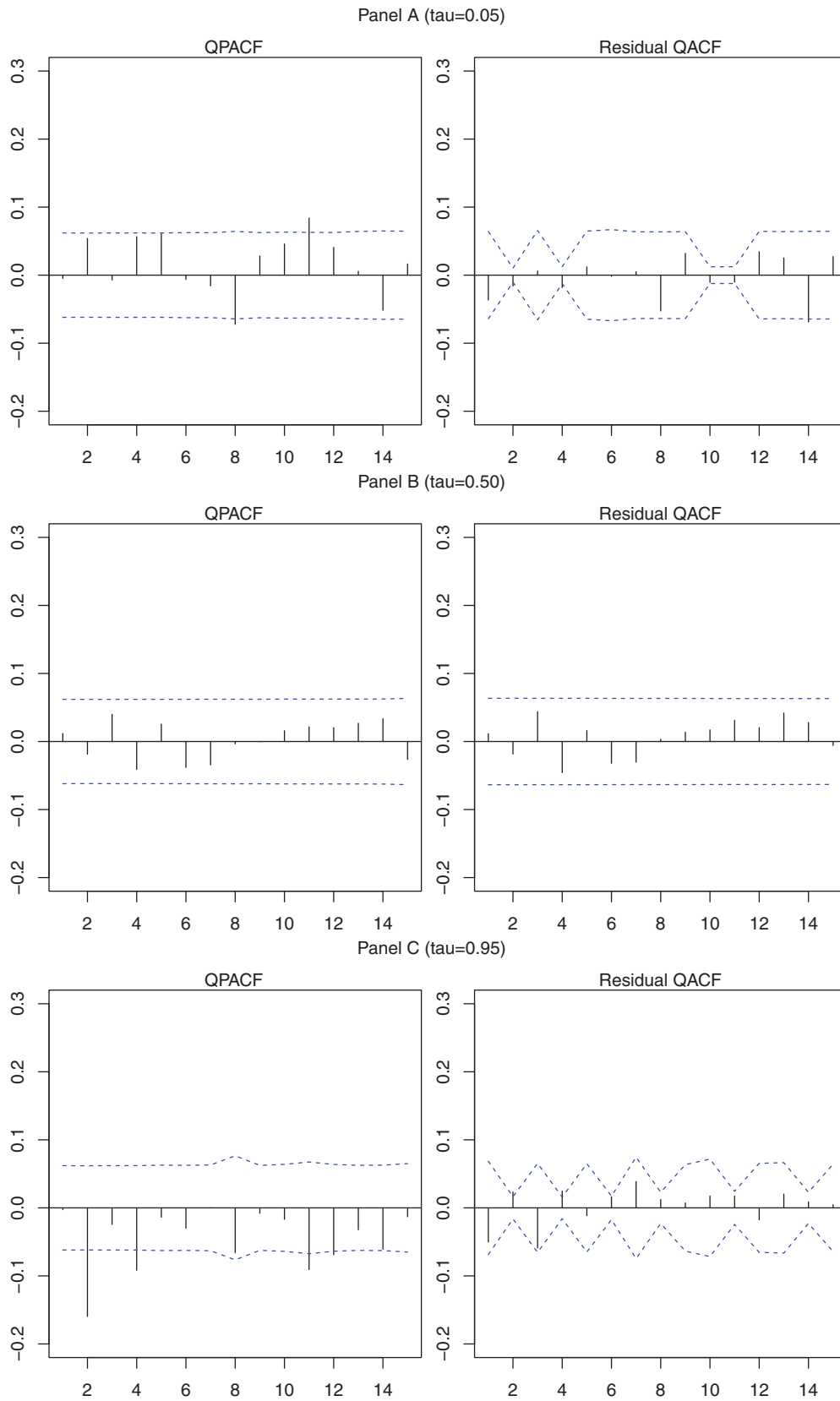
Figure 3. The sample QPACF of daily closing prices on the Nasdaq Composite and the sample QACF of residuals from the fitted models for $\tau = 0.05$, 0.5, and 0.95. The dashed lines in the left and right panels correspond to $\pm 1.96\sqrt{\widehat{\Omega}_3/n}$ and $\pm 1.96\sqrt{\widehat{\Omega}_5/n}$, respectively.
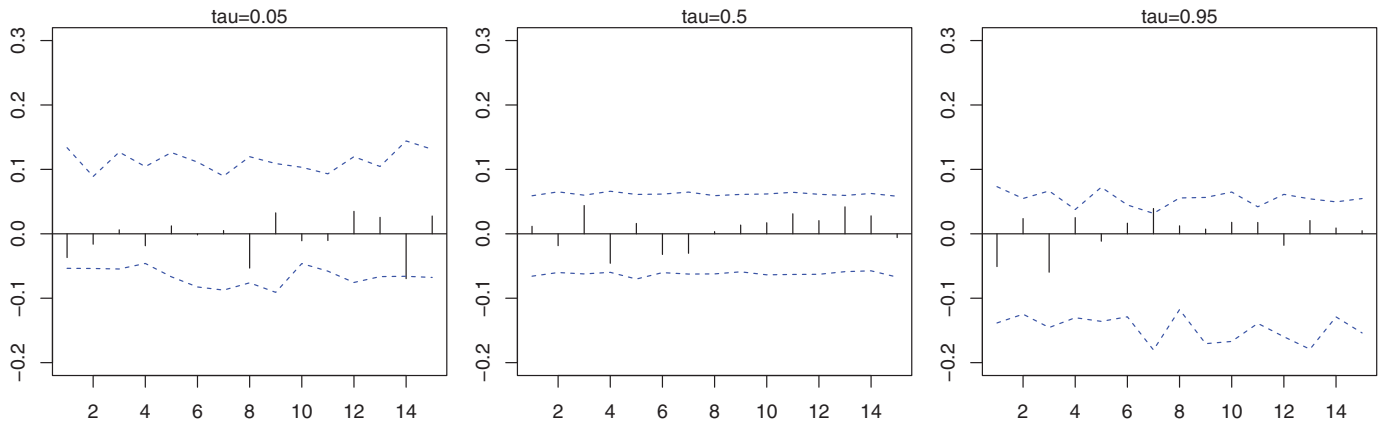
Figure 4. The sample QACF of residuals from the fitted models for $\tau = 0.05$, 0.5, and 0.95. The dashed lines correspond to 2.5th and 97.5th percentiles of the bootstrapped distributions, respectively.

We retain them, since the p-value of $Q_{\mathrm{BP}}(15)$ will be less than 0.06 if any one of them is deleted. Taking the above results as a whole, the model (5.1) is adequate.

We next consider the scenario with $\tau = 0.5$. The sample QPACF in Panel B indicates that all lags are insignificant. Hence, we fit the following model,

$$\widehat{Q}_{0.5}(y_t|\mathcal{F}_{t-1}) = 0.0467_{0.0133}. \qquad (5.2)$$

None of the lags in the sample QACF of residuals in Panel B of Figure 3 shows significance, and the p-value of $Q_{\mathrm{BP}}(15)$ is 0.750. Consequently, the above model is appropriate.

Finally, we study the upper quantile scenario with $\tau = 0.95$. The sample QPACF in Panel C exhibits that lags 2, 4, 11, 12, and 14 are significant. By Theorem 3, the QAR(14) model is suggested. We then employ the backward variable selection procedure to refine the model by removing lags $\{7, 10, 9, 5, 13, 12, 3, 1\}$ sequentially. The resulting model is

$$\begin{aligned}
\widehat{Q}_{0.95}(y_t|\mathcal{F}_{t-1}) = {} & 0.6809_{0.0233} - 0.2497_{0.0398}\, y_{t-2} \\
& - 0.1355_{0.0354}\, y_{t-4} - 0.0712_{0.0434}\, y_{t-6} \\
& - 0.1296_{0.0468}\, y_{t-8} - 0.1506_{0.0442}\, y_{t-11} \\
& - 0.1246_{0.0414}\, y_{t-14}, \qquad\qquad\qquad (5.3)
\end{aligned}$$

where all coefficients are significant at the 5% significance level, except that lag 6 is marginal. In addition, the p-value of $W_n(\tau)$ is 0.128, which demonstrates that the deleted coefficients are jointly insignificant. Although the QACF of residuals at lags 2 and 4 in Panel C of Figure 3 show marginal significance, the p-value of $Q_{\mathrm{BP}}(15)$ is 0.443 and it is not significant. In sum, the model (5.3) fits the data reasonably well. It is worth noting that the bootstrap results of the sample QACF of residuals in Figure 4 also indicate the model (5.3) as well as models (5.1) and (5.2) fitting the data adequately.

In addition to QPACF, we employ the Bayesian information criterion (BIC) considered by Koenker and Xiao (2006) to select the order of QAR models. Accordingly, the orders of 2, 0, and 7 are chosen at quantiles $\tau = 0.05$, 0.5, and 0.95, respectively. Furthermore, the Akaike information criterion (AIC) is applied, and the orders 2, 0, and 13 are correspondingly selected for quantiles $\tau = 0.05$, 0.5, and 0.95. At $\tau = 0.05$, both AIC and BIC choose lag 2, which stands out in the QPACF plot of Panel A and it is also included in model (5.1). However, the p-value

of $W_n(\tau)$ test for the QAR(2) model is 0.003. This may be due to missing lag 11, whose QPACF is significant and this lag is contained in model (5.1). It is of interest to note that AIC, BIC, and our proposed procedure yield the same model when $\tau = 0$. At the upper quantile with $\tau = 0.95$, AIC does not identify lag 14 and BIC does not choose lags 11 and 14. These two lags are significant via their QPACF measures and both are included in model (5.3). This may explain why the p-values of the $Q_{\mathrm{BP}}(15)$ test for the final models obtained via the backward selection procedure from QAR(7), chosen by BIC, and QAR(13), chosen by AIC, respectively, are less than 0.07. As mentioned in Box, Jenkins, and Reinsel (2008, p. 212), both AIC and BIC can be viewed as supplementary guidelines to assist in the model selection process. In addition, our purpose in this example is model fitting; hence, we conclude that models (5.1) to (5.3) are adequate. However, this does not exclude other possible models selected via different purposes or approaches.

Based on the three fitted QAR models, (5.1), (5.2), and (5.3), we obtain the following conclusions. (i) The lag coefficients at the lower quantile ($\tau = 0.05$) are all positive. This indicates that if the returns in past days have been positive (negative), then, when today's return is in the same direction, it is alleviated (even lower). It also implies that stock markets underreact to good news in bad times (i.e., $\tau = 0.05$). (ii) The lag coefficients at the upper quantile ($\tau = 0.95$) are all negative. This shows that if the returns in past days have been negative (positive), then, when today's return is in the different direction, it is even higher (dampened). As a result, stock markets overreact to bad news in good times (i.e., $\tau = 0.95$). (iii) As we expected, the intercept only at $\tau = 0.5$ shows no dependence for the conditional median of returns. Accordingly, Equation (5.2) indicates that today's return is not affected by the returns of recent past days. Although we only report the results of the lower and higher quantiles at $\tau = 0.05$ and $\tau = 0.95$, our studies yield the same conclusions across various lower and upper quantiles. In sum, our proposed methods support Veronesi's (1999) equilibrium explanation for stock market reactions.

6. DISCUSSION

In quantile regression models, we propose the quantile correlation and quantile partial correlation. Then, we apply them

to the quantile autoregressive model, which yields the quantile autocorrelation and quantile partial autocorrelation. In practice, the response time series may depend on exogenous variables. Hence, it is of interest to extend those correlation measures to the quantile autoregressive model with the exogenous variables given below:

$$Q_\tau(y_t|\mathcal{F}_{t-1}) = \phi_0(\tau) + \sum_{i=1}^{p} \phi_i(\tau)y_{t-i} + \beta'(\tau)\mathbf{x}_t,$$

$$\text{for} \quad 0 < \tau < 1,$$

where $\mathbf{x}_t$ is a vector of time series, and $\phi_i(\tau)$ and $\beta(\tau)$ are functions $[0, 1] \to R$; see Galvao, Montes-Rojas, and Park (2013). Following the definition of QPACF in Section 3.1, we can define $\phi_{kk,\tau} = \text{qpcor}_\tau\{y_t, y_{t-k}|\mathbf{z}_{t,k-1}, \mathbf{x}_t\}$. Accordingly, this allows us to extend our results in Section 3 to the above model.

In the context of growth charts, Wei and He (2006) considered a semiparametric quantile regression model,

$$y_j = g(t_j) + \sum_{l=1}^{p} \phi_l(t_j - t_{j-l})y_{j-l} + \beta'\mathbf{x}_j + e_j, \quad (6.1)$$

for n subjects, where each subject has measurements at random time $t_1, \ldots, t_m$, $g(\cdot)$ is a smooth function, the ϕ_l's are linear functions, and e_j is the random error with the τth quantile being zero; see Wei et al. (2006). This model can also be viewed as an extension of the QAR model. Then, let

$$(g_0, \phi_{01}, \ldots, \phi_{0,k-1}, \beta_0) = \text{argmin } E\Bigg[\rho_\tau\Bigg\{y_j - g(t_j)$$
$$- \sum_{l=1}^{k-1} \phi_l(t_j - t_{j-l})y_{j-l} - \beta'\mathbf{x}_j\Bigg\}\Bigg].$$

As a result, $y_j^* = g_0(t_j) + \sum_{l=1}^{k-1} \phi_{0l}(t_j - t_{j-l})y_{j-l} + \beta_0'\mathbf{x}_j$ is the effect of $\mathbf{z}_{j,k-1}$ and $\mathbf{x}_j$ on the τth quantile of y_j. Subsequently, we can define the QPACF as

$$\phi_{kk,\tau} = \text{qpcor}_\tau\{y_j, y_{j-k}|\mathbf{z}_{j,k-1}, \mathbf{x}_j\}$$
$$= \frac{\text{cov}\{\psi_\tau(y_j - y_j^*), y_{j-k}\}}{\sqrt{\text{var}\{\psi_\tau(y_j - y_j^*)\}\text{var}\{y_{j-k}\}}}. \quad (6.2)$$

This, together with the estimation method in Wei and He (2006), allows us to generalize our proposed procedure to this model.

The third possible generalization of the QAR model is the dynamic model with partially varying coefficients in Cai and Xiao (2012), which has a similar form to (6.1). Accordingly, we can obtain a measure analogous to that in (6.2). This enables us to extend our method to their model's identification and diagnostic checking. Clearly, the contribution of the proposed measures is not limited to the above three models. For example, the diagnosis of nonlinear quantile autoregression models (e.g., Chen, Koenker, and Xiao 2009) and quantile GARCH models (Xiao and Koenker 2009) can be considered; variable screening and selection (e.g., Fan and Lv 2008; Wang 2009) in quantile regressions is another important topic for future research. In

sum, this article introduces practical measures to broaden and facilitate the use of quantile models.

APPENDIX: TECHNICAL PROOFS

The appendix presents the technical proofs of Lemmas 1 and 2 and Theorems 1 and 3. Since the proofs of Theorems 2 and 4 are similar to those of Theorems 1 and 3, respectively, they are given in the online supplemental material.

Proof of Lemma 1. For $a, b \in R$, denote the function $h(a, b) = E[\rho_\tau(\varepsilon - a - bX)]$. It is known that $h(a, b)$ is a convex function with $\lim_{a^2+b^2\to\infty} h(a, b) = +\infty$. For $u \neq 0$,

$$\rho_\tau(u - v) - \rho_\tau(u)$$
$$= -v\psi_\tau(u) + \int_0^v [I(u \leq s) - I(u < 0)]ds$$
$$= -v\psi_\tau(u) + (u - v)[I(0 > u > v) - I(0 < u < v)], \quad \text{(A.1)}$$

see Knight (1998) and Koenker and Xiao (2006). Let $Y^* = \varepsilon - a - bX$. Then, the above equation, together with Hölder's inequality and the continuity of random variables X and ε, leads to

$$\left|\frac{1}{c}[h(a, b + c) - h(a, b)] + E[\psi_\tau(Y^*)X]\right|$$
$$= \left|\frac{1}{c}E[\rho_\tau(Y^* - cX) - \rho_\tau(Y^*)] + E[\psi_\tau(Y^*)X]\right|$$
$$= \left|\frac{1}{c}E\{(Y^* - cX)[I(0 > Y^* > cX) - I(0 < Y^* < cX)]\}\right|$$
$$\leq E[|X|I(|Y^*| < |c| \cdot |X|)] \leq (EX^2)^{1/2}[P(|Y^*|/|X| < |c|)]^{1/2},$$

which tends to zero as $c \to 0$. Accordingly,

$$\frac{\partial h(a, b)}{\partial b} = -E[\psi_\tau(\varepsilon - a - bX)X]. \quad \text{(A.2)}$$

Analogously, we have that

$$\frac{\partial h(a, b)}{\partial a} = -E[\psi_\tau(\varepsilon - a - bX)], \quad \text{(A.3)}$$

which is zero at $a = Q_{\tau, \varepsilon - bX}$. By Hölder's inequality and the continuity of random variables X and ε, we can further show that $\partial h(a, b)/\partial b$ and $\partial h(a, b)/\partial a$ are continuous functions.

Let $h_1(b) = h(Q_{\tau, \varepsilon - bX}, b)$. For any $b_1, b_2 \in R$ and $0 < w < 1$, by the convexity of $h(a, b)$, we have that

$$wh_1(b_1) + (1 - w)h_1(b_2)$$
$$= wh(Q_{\tau, \varepsilon - b_1 X}, b_1) + (1 - w)h(Q_{\tau, \varepsilon - b_2 X}, b_2)$$
$$\geq h(wQ_{\tau, \varepsilon - b_1 X} + [1 - w]Q_{\tau, \varepsilon - b_2 X}, wb_1 + [1 - w]b_2)$$
$$\geq h_1(wb_1 + [1 - w]b_2).$$

Accordingly, $h_1(b)$ is a convex function. Note that, under the conditions in Lemma 1, $Q_{\tau, \varepsilon - bX}$ is differentiable with respect to b. This, together with (A.2) and (A.3), implies

$$\frac{\partial h_1(b)}{\partial b} = -E[\psi_\tau(\varepsilon - Q_{\tau, \varepsilon - bX} - bX)X] = -\varrho(-b),$$

and it is a continuous and increasing function, by the convexity of $h_1(b)$. As a result, $\varrho(b)$ is a continuous and increasing function, which completes the first part of the proof.

Next, if $b = 0$, then $\varrho(0) = E[\psi_\tau(\varepsilon - Q_{\tau, \varepsilon})X] = 0$. Let $\xi = Q_{\tau, \varepsilon - bX} + bX$. Then, for any b such that $\varrho(b) = 0$, we have that

$$0 = h_1(b) - h_1(0) = E[\rho_\tau(\varepsilon - \xi) - \rho_\tau(\varepsilon)]$$
$$= -E[\xi\psi_\tau(\varepsilon)] + E[(\varepsilon - \xi)I(0 > \varepsilon > \xi)]$$
$$+ E[(\xi - \varepsilon)I(0 < \varepsilon < \xi)]$$
$$= E[(\varepsilon - \xi)I(0 > \varepsilon > \xi)] + E[(\xi - \varepsilon)I(0 < \varepsilon < \xi)].$$

Note that both $(\varepsilon - \xi)I(0 > \varepsilon > \xi)$ and $(\xi - \varepsilon)I(0 < \varepsilon < \xi)$ are non-negative random variables, and $\varepsilon - \xi$ is a continuous random variable. Thus, with probability one, $I(0 > \varepsilon > \xi) = I(0 < \varepsilon < \xi) = 0$, which yields $\xi = 0$. This implies that $b = 0$, and the proof is complete. □

Proof of Lemma 2. For $k = p$, let

$$(\alpha_{2,\tau}, \beta'_{2,\tau}) = \underset{\alpha,\beta}{\operatorname{argmin}}\, E[\rho_\tau(y_t - \alpha - \beta' \mathbf{z}_{t,p-1})]$$

and $y_t^* = y_t - \alpha_{2,\tau} - \beta'_{2,\tau}\mathbf{z}_{t,p-1}$. If $\phi_{pp,\tau} = 0$, then $\operatorname{qcov}\{y_t^*, y_{t-p}\} = 0$. Subsequently, applying the same techniques used in the proof of Lemma 1, we obtain that $(\alpha_{2,\tau}, \beta'_{2,\tau}, 0) = (\alpha_{3,\tau}, \beta'_{3,\tau}, \gamma_{3,\tau})$, where $(\alpha_{3,\tau}, \beta'_{3,\tau}, \gamma_{3,\tau}) = \operatorname{argmin}_{\alpha,\beta,\gamma} E[\rho_\tau(y_t - \alpha - \beta'\mathbf{z}_{t,p-1} - \gamma y_{t-p})]$. According to the definition of the QAR model (3.1), $(\alpha_{3,\tau}, \beta'_{3,\tau}, \gamma_{3,\tau}) = (\phi_0(\tau), \phi_1(\tau), \ldots, \phi_p(\tau))$, which implies that $\phi_p(\tau) = 0$. Since $\phi_p(\tau) \neq 0$, we have that $\phi_{pp,\tau} \neq 0$.

Let $e_{t,\tau} = y_t - \phi_0(\tau) - \phi_1(\tau)y_{t-1} - \cdots - \phi_p(\tau)y_{t-p}$. By (3.2), $I(e_{t,\tau} > 0)$ is independent of y_{t-k} for any $k > 0$. In addition, $(\alpha_{2,\tau}, \beta'_{2,\tau}) = (\phi_0(\tau), \phi_1(\tau), \ldots, \phi_p(\tau), \mathbf{0}')$ for $k > p$, where $\mathbf{0}$ is a $(k - p) \times 1$ vector with all elements being zero. Hence, $\phi_{kk,\tau} = 0$ for $k > p$. □

Proof of Theorem 1. For $u \neq 0$, we have that $I(u - v < 0) - I(u < 0) = I(v > u > 0) - I(v < u < 0)$. Using this result, we then obtain

$$\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X})$$
$$= \frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - Q_{\tau,Y})X_i + \frac{1}{n}A_n - \bar{X}\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y}),$$
$$(A.4)$$

where $A_n = \sum_{i=1}^{n} g_\tau(Y_i, Q_{\tau,Y}, \widehat{Q}_{\tau,Y})X_i$ and

$$g_\tau(Y_i, Q_{\tau,Y}, \widehat{Q}_{\tau,Y})$$
$$= \psi_\tau(Y_i - \widehat{Q}_{\tau,Y}) - \psi_\tau(Y_i - Q_{\tau,Y})$$
$$= -[I(Y_i < \widehat{Q}_{\tau,Y}) - I(Y_i < Q_{\tau,Y})]$$
$$= I(\widehat{Q}_{\tau,Y} - Q_{\tau,Y} < Y_i - Q_{\tau,Y} < 0)$$
$$\quad - I(\widehat{Q}_{\tau,Y} - Q_{\tau,Y} > Y_i - Q_{\tau,Y} > 0).$$

It can be shown that

$$\left|\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})\right| = \left|\tau - \frac{1}{n}\sum_{i=1}^{n}I(Y_i - \widehat{Q}_{\tau,Y})\right|$$
$$= \left|\tau - \frac{[n\tau]}{n}\right| \leq \frac{1}{n}.$$

This, together with the law of large numbers, implies the last term of (A.4) satisfying

$$\bar{X}\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y}) = O_p(n^{-1}). \qquad (A.5)$$

We next consider the second term on the right-hand side of (A.4). For any $v \in R$, denote

$$\xi_n(v) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\{g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y} + n^{-1/2}v)$$
$$- E[g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y} + n^{-1/2}v)|X_i]\}X_i,$$

where $E[g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y} + n^{-1/2}v)|X_i] = -\int_{Q_{\tau,Y}}^{Q_{\tau,Y}+n^{-1/2}v} f_{Y_i|X_i}(y)dy$ and $f_{Y_i|X_i}(\cdot)$ is the conditional density of Y_i given X_i. Then, by Hölder's inequality, we have that

$$E[\xi_n(v)]^2 = E[g_\tau(Y_i, Q_{\tau,Y}, Q_{\tau,Y} + n^{-1/2}v)X_i]^2$$
$$\leq [P(|Y_i - Q_{\tau,Y}| < n^{-1/2}v)]^{1/2}[EX_i^4]^{1/2} = o(1). \quad (A.6)$$

After algebraic simplification, we further obtain

$$\sup_{|v_1-v|<\delta}|\xi_n(v_1) - \xi_n(v)|$$
$$\leq \sup_{|v_1-v|<\delta}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\{g_\tau(v_1) - g_\tau(v)\}X_i|$$
$$\quad + E[|\{g_\tau(v_1) - g_\tau(v)\}X_i||X_i]$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\{g_\tau(v_1^*) - g_\tau(v)\}X_i| + E[|\{g_\tau(v_1^*) - g_\tau(v)\}X_i||X_i],$$

where v_1^* takes the value of $v + \delta$ or $v - \delta$. Hence,

$$E\sup_{|v_1-v|<\delta}|\xi_n(v_1) - \xi_n(v)|$$
$$\leq 2\sqrt{n}E|\{g_\tau(v_1^*) - g_\tau(v)\}X_i|$$
$$= 2\sqrt{n}E\left|\int_{Q_{\tau,Y}+n^{-1/2}v}^{Q_{\tau,Y}+n^{-1/2}v_1^*} f_{Y_i|X_i}(y)dyX_i\right|$$
$$\leq \delta\cdot 2E\left[\sup_{|y|\leq\pi} f_{Y_i|X_i}(Q_{\tau,Y} + y)|X_i|\right], \qquad (A.7)$$

where $|n^{-1/2}v| < \pi$ and $|n^{-1/2}v_1^*| < \pi$ when n is large. Both (A.6) and (A.7), in conjunction with the theorem's assumptions and the finite converging theorem, imply that $E\sup_{|v|\leq M}|\xi_n(v)| = o(1)$ for any $M > 0$. In addition, applying the theorem in Section 2.5.1 of Serfling (1980), we have

$$\sqrt{n}(\widehat{Q}_{\tau,Y} - Q_{\tau,Y}) = f_Y^{-1}(Q_{\tau,Y})\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_\tau(Y_i - Q_{\tau,Y}) + o_p(1)$$
$$= O_p(1).$$

Accordingly,

$$\frac{1}{\sqrt{n}}A_n = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{Q_{\tau,Y}}^{Q_{\tau,Y}+(\widehat{Q}_{\tau,Y}-Q_{\tau,Y})} f_{Y_i|X_i}(y)dyX_i + o_p(1)$$
$$= -(\widehat{Q}_{\tau,Y} - Q_{\tau,Y})\frac{1}{\sqrt{n}}\sum_{i=1}^{n}f_{Y_i|X_i}(Q_{\tau,Y})X_i + o_p(1)$$
$$= -\frac{E[f_{Y_i|X_i}(Q_{\tau,Y})X_i]}{f_Y(Q_{\tau,Y})}\cdot\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\psi_\tau(Y_i - Q_{\tau,Y}) + o_p(1).$$
$$(A.8)$$

Subsequently, using (A.4), (A.5), and (A.8), we obtain that

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) - \operatorname{qcov}_\tau\{Y, X\}\right]$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) - \operatorname{qcov}_\tau\{Y, X\}]$$
$$= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\psi_\tau(Y_i - Q_{\tau,Y})(X_i - \mu_{X|Y}) - \operatorname{qcov}_\tau\{Y, X\}] + o_p(1),$$
$$(A.9)$$

where $\mu_{X|Y}$ is defined in Section 2.2. Since

$$\sqrt{n}(\bar{X} - \mu_X)^2 = \frac{1}{\sqrt{n}}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu_X)\right]^2 = O_p(n^{-1/2}),$$

we further have that

$$\sqrt{n}(\widehat{\sigma}_X^2 - \sigma_X^2) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}[(X_i - \mu_X)^2 - \sigma_X^2] + o_p(1). \quad (A.10)$$

Moreover, (A.9), (A.10), the central limit theorem, and the Cramer–Wold device, lead to

$$\sqrt{n}\left(\begin{array}{c}\widehat{\sigma}_X^2 - \sigma_X^2 \\ n^{-1}\sum_{i=1}^{n}\psi_\tau(Y_i - \widehat{Q}_{\tau,Y})(X_i - \bar{X}) - \operatorname{qcov}_\tau\{Y, X\}\end{array}\right) \to_d N(0, \Sigma),$$

where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{13} & \Sigma_{12} \end{pmatrix},$$

and Σ_{11}, Σ_{12}, and Σ_{13} are defined in Section 2.2. Finally, following the Delta method (van der Vaart 1998, chap. 3), we complete the proof. □

Proof of Theorem 3. We first consider the term $\widetilde{\sigma}_{y|\mathbf{z}}^2$ in $\widetilde{\phi}_{kk,\tau}$. Let $\mathbf{z}_{t,k-1}^* = (1, \mathbf{z}_{t,k-1}')'$. Since $E y_t^2 < \infty$ and $E[y_t - E(y_t|\mathcal{F}_{t-1})]^2 > 0$, the matrix $E(\mathbf{z}_{t,k-1}^* \mathbf{z}_{t,k-1}^{*\prime})$ is finite and positive definite. We then can show that

$$\widetilde{\sigma}_{y|\mathbf{z}}^2 = \frac{1}{n} \sum_{t=k+1}^{n} (y_{t-k} - \alpha_1 - \beta_1' \mathbf{z}_{t,k-1})^2 + o_p(n^{-1/2})$$

$$= E(y_{t-k} - \alpha_1 - \beta_1' \mathbf{z}_{t,k-1})^2 + o_p(1). \quad (A.11)$$

We next study the numerator of $\widetilde{\phi}_{kk,\tau}$. Let $\theta_{2,\tau} = (\phi_0(\tau), \phi_1(\tau), \dots, \phi_p(\tau), \mathbf{0}')'$, and $\widetilde{\theta}_{2,\tau} = (\widetilde{\alpha}_{2,\tau}, \widetilde{\beta}_{2,\tau}')'$, where $\mathbf{0}$ is the $(k-p) \times 1$ vector defined in the proof of Lemma 2, and $\widetilde{\alpha}_{2,\tau}$ and $\widetilde{\beta}_{2,\tau}$ are defined in Section 3.1. It is noteworthy that the series $\{y_t\}$ is fitted by model (3.1) with order $k-1$ and the true parameter vector $\theta_{2,\tau}$. Accordingly, $e_{t,\tau} = y_t - \theta_{2,\tau}' \mathbf{z}_{t,k-1}^*$ and the parameter estimate of $\theta_{2,\tau}$ is $\widetilde{\theta}_{2,\tau}$. Then, from the proof of Theorem 4, we obtain that

$$\sqrt{n}(\widetilde{\theta}_{2,\tau} - \theta_{2,\tau}) = \{E[f_{t-1}(0)\mathbf{z}_{t,k-1}^* \mathbf{z}_{t,k-1}^{*\prime}]\}^{-1}$$
$$\cdot \frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau}) \mathbf{z}_{t,k-1}^* + o_p(n^{-1/2}).$$

Applying a similar approach to that used in obtaining (A.8), and then using the above result, we further have that

$$\frac{1}{n} \sum_{t=k+1}^{n} [\psi_\tau(y_t - \widetilde{\theta}_{2,\tau}' \mathbf{z}_{t-k}^*) - \psi_\tau(e_{t,\tau})] y_{t-k}$$

$$= -\frac{1}{n} \sum_{t=k+1}^{n} \int_0^{(\widetilde{\theta}_{2,\tau} - \theta_{2,\tau})' \mathbf{z}_{t,k-1}^*} f_{t-1}(s) ds \, y_{t-k} + o_p(n^{-1/2})$$

$$= -(\widetilde{\theta}_{2,\tau} - \theta_{2,\tau})' \cdot \frac{1}{n} \sum_{t=k+1}^{n} f_{t-1}(0) y_{t-k} \mathbf{z}_{t,k-1}^* + o_p(n^{-1/2})$$

$$= -A_1'(\tau) \Sigma_{31}^{-1}(\tau) \cdot \frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau}) \mathbf{z}_{t,k-1}^* + o_p(n^{-1/2}), \quad (A.12)$$

where $A_1(\tau)$ and $\Sigma_{31}(\tau)$ are defined as in Section 3.1. Subsequently, using similar techniques to those for obtaining (A.4) and the result from Equation (A.12), we obtain that

$$\frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(y_t - \widetilde{\alpha}_{2,\tau} - \widetilde{\beta}_{2,\tau}' \mathbf{z}_{t,k-1}) y_{t-k}$$

$$= \frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau}) y_{t-k} + \frac{1}{n} \sum_{t=k+1}^{n} [\psi_\tau(y_t - \widetilde{\theta}_{2,\tau}' \mathbf{z}_{t,k-1}^*) - \psi_\tau(e_{t,\tau})] y_{t-k}$$

$$= \frac{1}{n} \sum_{t=k+1}^{n} \psi_\tau(e_{t,\tau})[y_{t-k} - A_1'(\tau) \Sigma_{31}^{-1}(\tau) \mathbf{z}_{t,k-1}^*] + o_p(n^{-1/2}). \quad (A.13)$$

Subsequently, applying a method similar to the proof of Theorem 2.1 in Li and Li (2008) and Gutenbrunner and Jureckova (1992), we can show that the left-hand side of (A.13) is tight. This, in conjunction with Equation (A.11), the central limit theorem for the martingale difference sequence, the Cramer–Wold device, and Theorem 7.1 in Billingsley (1999), completes the proof of theorem. From Lemma 2, we also have that $\phi_{kk,\tau} = 0$. □

SUPPLEMENTARY MATERIALS

The supplementary materials consist of the detailed technical proofs of Theorems 2, 4, and 5 as well as three detailed bootstrap algorithms from Section 3.4. We also provide additional simulation results for studying the proposed measures and tests in Sections 2 and 3 of the article.

[Received December 2012. Revised January 2014.]

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